TEACHERS' GUIDE
IN
MATHEMATICS FOR CLASS XI

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C O N T E N T S

FOREWORD

The Department of Education in Science and Mathematics has been working in the area of curriculum development in science and mathematics since a long time. After the publication of the National Policy on Education (1986), a new mathematics curriculum has been developed with the active collaboration of mathematics educators from all parts of the country. To achieve the desired goals of Mathematics education, the Department is bringing out not only the textbooks but the entire instructional package consisting of Problem books/enrichment materials, Teachers' Guides and other materials.

The new textbook of mathematics for Class XI was already published. It was first introduced in all Kendriya Vidyalayas during the year 1988-89. All other schools affiliated to CBSE, introduced the book from 89-90. Several other Boards of Education in the country are adopting/adapting this new curriculum & textbook and these would be in use in the coming years in many parts of the country.

Teacher's Guide is an essential component of the instructional package and an instrument for effective classroom teaching/learning.

The first draft of the materials for this teachers' guide was developed in a workshop held at The Ramanujan Institute for advanced study in mathematics, University of Madras, Madras from August 21 to 27, 1989 under the guidance of Prof. M.S.Rangachari who was also a member of the NCERT's writing team. Later the draft materials were finalised

in a workshop held at SCERT Udaipur from 15.1.90 to 24.1.90. We are aware that there is still scope to improve these materials. We thought it would be more useful to circulate this materials among a larger group of teachers and teacher educators of mathematics and get their valuable suggestions so that these suggestions could be incorporated in the final printing edition. The Department of Education in science and mathematics will be grateful to all those who would favour us with their valuable suggestions/comments for the improvement of these draft materials and their contributions will be acknowledged.

I am grateful to all the participants of the two workshops for their valuable contributions, whose names are appearing separately. I am thankful to my colleagues in the Department Prof. K.V. Rao, Prof. S.C.Das, Dr. B. Deokinandan, Dr. Hukum Singh and Dr. Mukti Achary who took active part in the discussion on various topics in the worksho This book would not have com e in the present form but for the untiring efforts and hard work of Dr. Hukum Singh, who coordinated the whole activity. I am very thankful to him.

New Delhi.

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CHAPTER - I

SETS, RELATIONS AND FUNCTIONS

1.1 Introduction

The concept of sets, relations and functions serve as a fundamental part of the present day Mathematics. The study of sets have already been carried out in class IX. The main aim of this chapter is to study the sets, relations and functions in detail as these concepts are important for the study of advanced mathematics. Sets are used to define the concepts of relation and functions. The study of sequences, analytical geometry etc. requires the knowledge of sets. Sets open a new era in generalising certain concepts and their application.

In a nutshell, this chapter forms the foundation for modern mathematical knowledge and yet the concept is so simple and primitive that it does not contradicts anything that our students studied earlier in the previous classes.

G. Cantor (1845-1918) was the first who introduced and used the set theory in mathematics. He developed an intuitively grounded general theory of sets, treating particularly those sets having intuitively many members. R. Dedekind defined real numbers on the basis of sets of rational numbers. For example, if we divide the rational numbers into two sets such that one set, say L, contains all the numbers which are not greater than 2 and the other set, say R, contains all the

members which are greater than 2, then it is interesting to note that R has no least member while L has the greatest member 2. According to Dedekind, this division defines the number '2'. In case L (Lower class) has no greatest member and R (Upper class) has no least member, the section made by the sets L and R, defines an irrational number. The sections of this type of rational numbers made by the sets L and R where

(i) P_1 and P_2 are two mutually exclusive properties,

(ii) One of these is necessarily possessed by every rational number,

and

(iii) L has all the rational numbers having property P_1 and R has all the rational numbers having property P_2, are called Dedekind Cuts'.

1.2 Content Analysis

In this section the number of each subsection is in accordance with that in the textbook.

1.1 Sets

By a set, we mean a collection of objects which are definite and distinguishable. The objects in a set can be anything, viz., boys, girls, people, letters, rivers, numbers etc. These objects are called members or elements of the set. The important thing about the set is that it is considered to be a 'whole' i.e., an entity in itself. Cantor defined the set as

'A set is a collection of definite, distinguishable objects of our intuition or of our intellect to be conceived as a whole'.

It is important to remember that if the objects are not well defined, they do not form a set. For example, 'the collection of delicious dishes' is not well defined and thus does not form a set because a certain dish may be delicious according to one person and not so according to others. Similarly collection of good students is not a set and collection of tall persons in the world is not a set.

The concept of a subset of a set and the operations of union and intersection have been defined and discussed in the text book at length. It may be noted here that '$A \subset B$' implies that 'each member of A is also a member of B'. It may happen sometimes that $A = B$ and in that case also, $A \subset B$. However, if the teacher wishes he may discuss the concept of 'Proper Inclusion'. In that case $A \subset B$ iff every member of A is a member of B and there is at least one member of B which is not a member of A. If we are interested in proper inclusion, then the symbol '$\subseteq$' is used for a subset and '$\subset$' for a proper subset.

The set containing no elements is called the null (or empty) set and it is denoted by $\emptyset$. It should be noted here that

(i) null set is a subset of every set,

(ii) every set is a subset of itself.

The students often fail to see the reasons behind these two theorems. The logic for these may be given as follows:

(i) If A is a subset of B, there are no elements in A which do not belong to B. Since there are no elements in ϕ which do not belong to given set, ϕ is a subset of every sets.

(ii) Since all the elements of a given set A belong to itself, every set is a subset of itself.

When we talk of any subset of a given set A, naturally the curiosity arises to know how many subsets of a given set can be formed. It is important to note that we denote ' the set of all subsets of A' by P(A). P(A) is called the Power set of A. The following theorem which gives the solution of problem 11 in Exercise 1.1, is of great significance.

Theorem (1.1)

If a set A has n elements, then the set P(A) of all subsets of A has 2^n elements.

Proof: n is always a non-negative integer. Therefore there are

$^nC_1$ subsets each containing only one element;

$^nC_2$ subsets each containing only two elements;

$^nC_3$ subsets each containing only three elements;

....

....

^nC_n subsets i.e., one subset containing n elements

Lastly we note that $\emptyset$ is also one subset of A. Thus the total number of all subsets of A

$$= 1 + {}^nC_1 + {}^nC_2 + \cdots\cdots + {}^nC_n$$

$$= {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots\cdots + {}^nC_n$$

$$= (1 + 1)^n = 2^n$$

Difference of Sets

The difference of sets A and B is the set of elements which belong to A but which do not belong to B. We denote the difference of A and B by $A \setminus B$ which is read as 'A difference B' or simply ' A minus B '.

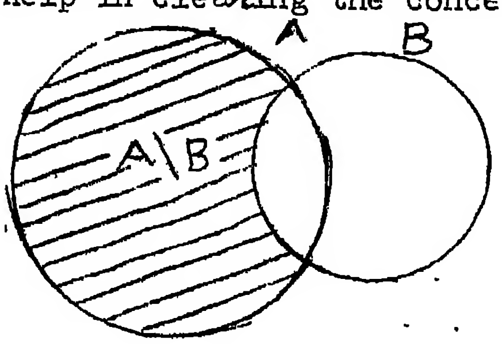
Thus

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$

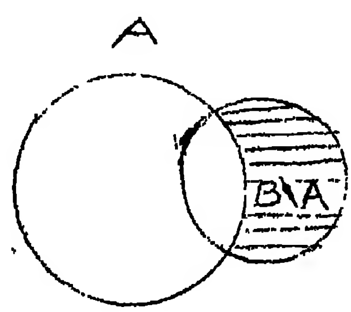
It may be noted that $A \setminus B \neq B \setminus A$ leaving aside the particular case when $A = B$, because

$$B \setminus A = \{y : y \in B \text{ and } y \notin A\}$$

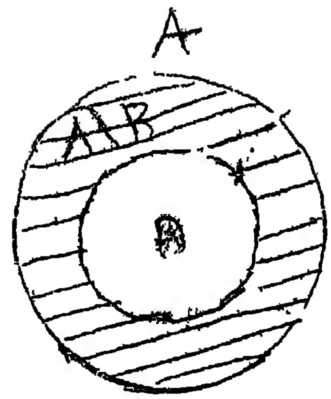
The following Venn diagrams related to difference sets help in clearing the concept



$A \setminus B$ is shaded
Fig. 1.1

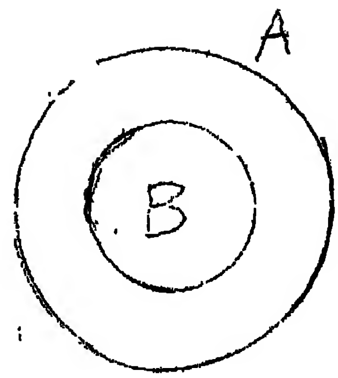


$B \setminus A$ is shaded
Fig. 1.2



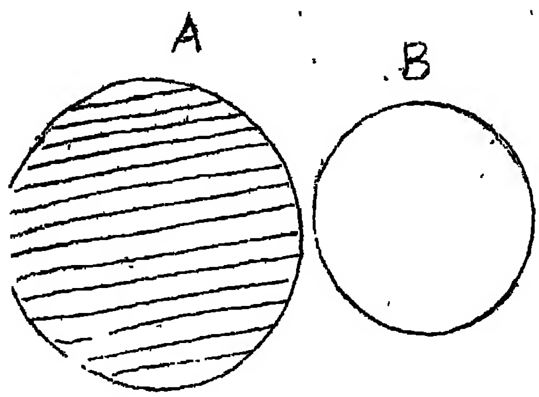
$A \setminus B$ is shaded
(Case: $B \subset A$)

Fig.1.3



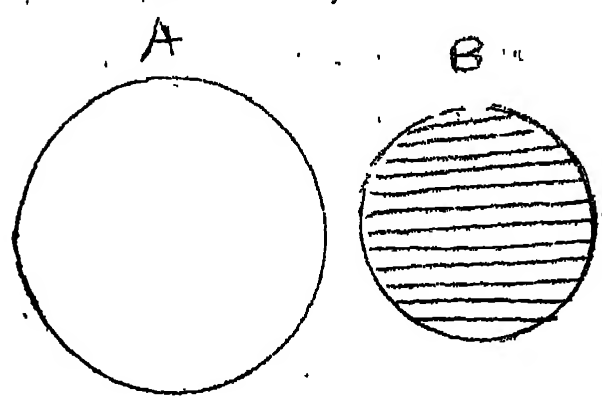
$B \setminus A$ is shaded
(Since no portion is shaded, it implies $B \setminus A = \emptyset$ when $B \subset A$)

Fig.1.4



$A \setminus B$ is shaded
(when A and B are disjoint sets,
$A \setminus B = A$)

Fig.1.5



$B \setminus A$ is shaded
(when A and B are disjoint sets, $B \setminus A = B$

Fig.1.6.

Remark

It is easily seen that

(i) $A \setminus B \subset A$ because $a \in A \setminus B \Rightarrow a \in A$

(ii) $B \setminus A \subset B$ because $b \in B \setminus A \Rightarrow b \in B$

(iii) $A \setminus B$, $B \setminus A$ and $A \cap B$ are disjoint

Sets as is obvious from the following Venn Diagram.

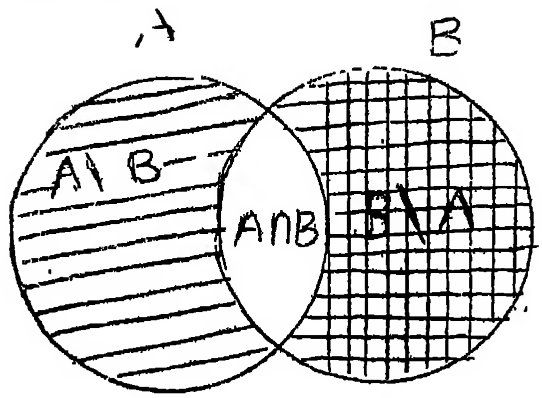


Fig.1.4

1.2 Cartesian Product of Sets, Relations

As a result of operations of union, intersection and difference of two sets, we got another set. Now we explain how we can get another set, called the cartesian product of two sets, with the help of the ordered pairs. The ordered pair of two elements a and b is denoted by (a,b). This is called ordered pair because the order in which we write these elements is important. So the ordered pair (a,b) is not the same as (b,a) because the order of elements is not the same in both the pairs. In fact, the order in one pair is reverse of that in the other. Thus the basic property of ordered pairs is that $(x,y) = (a,b) \Rightarrow x = a,\ y=b,$

and so $(a,b) = (b,a) \Rightarrow a=b.$

In the ordered pair (a,b), a is called the first member and b the second member.

The cartesian product of two sets A and B is the set of all possible ordered pairs such that the first member of each pair belongs to A and the second member of each pair belongs to B. The cartesian product is written as A x B :

$$A \times B = \{(a,b) : a \in A, b \in B\}$$

If $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3\}$

$$\text{then } A \times B = \{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_2, b_1), (a_2, b_2), (a_2, b_3)\}$$

$$\text{Also } A \times A = \{(a_1, a_1), (a_1, a_2), (a_2, a_1), (a_2, a_2)\}$$

Similarly, the cartesian product of two sets B and A, i.e., B x A will be the set of all possible ordered pairs whose first and second members belong to B and A respectively.

Let $|S|$ denote the number of elements in a set S (The number $|S|$ is called the cardinality of the set S). Now we see that for a particular element 'a' of A, the number of ordered pairs in A x B having a as its first member is $|B|$. Since there are $|A|$ different first members among the ordered pairs being considered, it follows that the total number of ordered pairs in

A x B is $|A|\ |B|$ So

$$|A \times B| = |A|\ |B| ,$$

i.e. ,cardinality of A x B is equal to the product of the cardinalities of A and B.

Relation.

Every set of ordered pairs represents some relation. If this set of ordered pairs is a subset of A x B, it is said to be a relation from A to B. It may be noted that every relation is a relation in some product set, say A x B. We call A x B the universe of discussion when we are discussing relations from A to B.

Example

1. Let $A = \{0, 1, 2\}$ and $B = \{3, 4\}$

Then

$$A \times B = \{(0,3), (0,4), (1,3), (1,4), (2,3), (2,4)\}$$

Since $R = \{(0,3), (2,4)\}$ is a subset of A x B, R is a relation. The domain of R is $\{0, 2\}$ and its range is $\{3, 4\}$. It may be noted that the set of the first elements of all ordered pairs in R is called the domain of R and the set of all second elements of these ordered pairs is called the range of R. Thus for $R : A \rightarrow B$,

Domain $R \subset A$,

Range $R \subset B$,

2. If $X = Y = R$ (set of all real numbers), we may define a relation r on R by setting

$r = \{(x,y) ; x \, r \, y \text{ iff } x < y\}$

Thus $(4,5) \in r$ since $4 < 5$ and so on. Remember $(5,4) \notin r$ since 5 is not less than 4. Thus this relation r is a proper subset of $R \times R$.

3. Let X be any set $P(X)$ is the set of all subsets of X. A relation R between X and $P(X)$ may be defined as

$$R = \{ (x,A) \in X \times P(X) \; ; \; x \in A, \quad A \subset P(X) \}$$

This relation R is the relation of ' belonging' between elements of X and elements of $P(X)$ such that if $x \in X$ and $A \in P(X)$ then $x \, RA \Leftrightarrow x \in A$.

4. Let $R = \{ (m,n) \in N \times N : m = 5n \}$

Then $R \subset N \times N$. Therefore R is a relation on the set N. We write $25 \, R5$, $30 \, R6$, $45 \, R9$, $45 \not R \, 8$, $8 \not R \, 48$.

Inverse of a Relation

Let R be a relation from a set A to a. Set B. The relation R^{-1} from B to A is said to be the inverse relation of R iff

$$R^{-1} = \{ (y,x) \; ; \; (x,y) \in R \}$$

Example

Let $A = \{ 1, m, n \}$ and $B = \{ a, b \}$

If we take relation

$$R = \{ (1, a), (1,b), (m, a) \; (n,b) \}$$

from A to B, then relation

$R^{-1} = \{ (a,l), (b,l), (a,m), (b,n) \}$

from B to A is the inverse of R.

Remark

Every relation has an inverse relation.

Equivalence Relation

A relation R from A to A is an equivalence relation if

(i) R is reflexive, that is, for every $a \in A$, $(a,a) \in R$, i.e., $a R a \ \forall \ a \in R$

(ii) R is symmetric, that is, $(a,b) \in R$ implies $(b,a) \in R$, i.e., $a R b \Rightarrow b R a$

(iii) R is transitive, that is, $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$, i.e., $a R b$, $b R c \Rightarrow a R c$.

Example

Let T be the set of all triangles in the Euclidean plane. Let R be a relation defined in T by 'x is similar to y '. Then R is an equivalence relation because it is

(i) reflexive since every triangle is similar to itself.

(ii) symmetric because if triangle x is similar to y, then y must be similar to x.

(iii) transitive because if triangle x is similar to y and y is similar to z, then triangle x is similar to z.

All the following relations are equivalence relation in the indicated sets,

(i) " Is in the same class" in the set of students in a school.

(ii) " Is parallel to " in the set of lines in the same plane.

(iii) " Has the same modules as " in the set of complex numbers.

(iv) " Live in the same house as " in the set of persons in a house.

(v) " Has the same number of pages " on the set of all books.

Following relations are not equivalence relation.

(i) " Is a brother/sister of " on the set of brothers/sisters in a family. For, this relation is not reflexive.

(ii) ' Is a husband/wife of " on the set of husbands/wives. For this relation is not symmetric.

One of the most important properties of the equivalence relations deal with what are called " <u>partitions of sets</u>" . This is explained below.

Let R be an equivalence relation on a set A. Then the set

$\left\{ x : x \, R \, a , x \in A, a \in A \right\}$ is a subset of A and is called an <u>equivalence class</u> of A determined by an element a of A. The equivalence class determined by an element $a \in A$ is denoted by $\bar{a}$. Thus $\bar{a} = \left\{ x : x \, R \, a, x \in A \right\}$.

Similarly we define $\bar{b}$, $\bar{c}$, ... where b, c ,.... $\in$ A.

The family consisting of all equivalence classes $\bar{a}, \bar{b}, \bar{c}, \ldots$ of A is called the <u>quotient set</u> of A w.r.t. the equivalence relation R and is denoted by $\bar{A}$. Thus $\bar{A} = \{ \bar{a}, \bar{b}, \bar{c}, \ldots \}$

Partition of a Set

Let B_r ($\subset A$) be a non-empty set for each value of r in an index set Δ. Then a family of non-empty sets $\{ B_r \}_{r \in \Delta}$ is called a partition of A if

(i) $A = \bigcup_{r \in \Delta} B_r$

(ii) For any $B_r , B_s \in \{ B_r \}_{r \in \Delta}$ either $B_r = B_s$ or $B_r \cap B_s = \emptyset$.

Example

Let $A = \{ 5,7,9,11,13,15,17,19 \}$

$B_1 = \{ 5,7,9 \}$, $B_2 = \{ 11, 13 \}$, $B_3 = \{ 15,17,19 \}$

$B_4 = \{ 11, 13 \}$, $\Delta = \{ 1,2,3,4 \}$.

Then the sets B_1, B_2, B_3 and B_4 have the following properties:

(i) $A = \bigcup_{r=1}^{4} B_r$ (ii) $B_r \cap B_s = \emptyset$ (or $B_r = B_s$)

for $r \neq s$. Evidently the family of sets $\{ B_r \}_{r \in \Delta}$ determines a <u>partition of the set</u> A. With these definitions in mind, we can now present the, following result.

<u>Theorem</u>

Let R be an equivalence relation in a set A, then the set $S = \{\bar{a} : a \in A\}$ of equivalence classes is a partition of the set A.

<u>Proof</u>

In order to prove the theorem, we must show that the set of equivalence classes satisfies the definition of partition of A. First of all, we observe that if $\bar{c}$ is an equivalence class, then $\bar{c}$ is non-empty since $c \in \bar{c}$. Furthermore, for an arbitrary element $b \in A$, we have $b \in \bar{b}$; therefore, the union of the elements of S is A.

Now it remains to show that every two distinct elements of S are disjoint. Suppose $\bar{b}$ and $\bar{c}$ are two disjoint elements of S. We shall assume that $\bar{b}$ and $\bar{c}$ are not disjoint. Then under this assumption we shall prove that $\bar{b} = \bar{c}$ which will contradict that $\bar{b}$ and $\bar{c}$ are distinct, implying that our assumption that $\bar{b}$ and $\bar{c}$ are not disjoint is incorrect.

Now if $\bar{b}$ and $\bar{c}$ are not disjoint, it follows that $\bar{b} \cap \bar{c} = \emptyset$ which implies that there exists at least one element $x \in A$ such that $x \in \bar{b} \cap \bar{c}$. Hence $x \in \bar{b}$ and $x \in \bar{c}$. Thus $x R b$ and $x R c$. Since R is symmetric, therefore $b R x$ and we have $b R x$ and $x R c$. Since R is transitive, we get $b R c$ and so $c R b$ also.

We now show that $\bar{b} = \bar{c}$. We do this by verifying that each of $\bar{b}$ and $\bar{c}$ is a subset of the other. Suppose

$y \in \bar{b} \Rightarrow y \; R \; b$ and we have shown that $b \; R \; c$, therefore $y \; R \; c$, since R is transitive.

$$\Rightarrow y \in \bar{c}$$
$$\Rightarrow \bar{b} \subset \bar{c}$$

Similarly if $d \in \bar{c} \Rightarrow d \; R \; c$

$$\Rightarrow d \; R \; b \quad (\because \; c \; R \; b)$$
$$\Rightarrow d \in \bar{b}$$
$$\Rightarrow \bar{c} \subset \bar{b}$$

Hence $\bar{b} = \bar{c}$

This proves the theorem.

The following example illustrates the above theorem.

Let $A = \{ 1,2,3,4,5,6 \}$ and let R be a relation such that

$$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4), (4,5), (5,4), (5,5), (6,6)\}.$$

It is clear that R is an equivalence relation in A. Hence according to the above theorem $\{\bar{a} \,.\, a \in A\}$ is a partition of A. Here the choices for 'a' are 1,2,3,4,5 and 6. We have $\bar{1} = \{1, 2, 3\}$, $\bar{2} = \{1, 2, 3\}$

$\bar{3} = \{1, 2, 3\}$, $\bar{4} = \{4,5\}$ $\bar{5} = \{4,5\}$

$\bar{6} = \{5\}$. Hence

$$\{\bar{a} : a \in A\} = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}$$
$$= \{\bar{1}, \bar{4}, \bar{6}\},$$

the last equality following because $\bar{1} = \bar{2} = \bar{3}$ and $\bar{4} = \bar{5}$.

It is clear that $\{\bar{1}, \bar{4}, \bar{6}\}$ is a partition of A. This is

shown by the figure below where we see howthe set A is partitioned by the equivalence relation R.

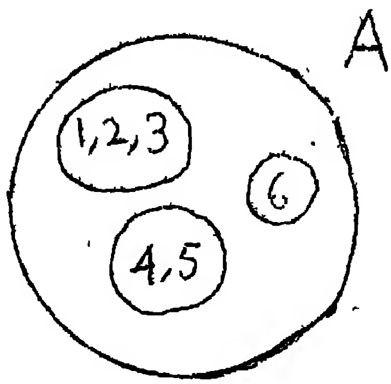


Fig. 1.10

Partial Order Relation

First we will define what is an anti-symmetric relation.

A relation in a set A is said to be antisymmetric if

$x, y \in A,\quad x\, R\, y$ and $y\, R\, x \Rightarrow x = y$.

For example, in the set of natural numbers N, the relation ' is divisor of ' is anti symmetric because a/b and b/a will only be true when a = b.

In the set of all sets, the relation 'is a subset of' is anti symmetric.

A relation which is (i) reflexive (ii) anti symmetric and (iii) transitive is called a partial order relation.

1.3 Functions

Let X and Y be two given non-empty sets. A relation f from X to Y is called a function from X to Y written as $f : X \rightarrow Y$ iff

(i) the domain of the relation f is the whole set X, i.e. each element of X is related to some element of Y.

(ii) for each $x \in X$, there is exactly one ordered pair $(x,y) \in f$, i.e. if $(x, y_1) \in f$ and $(x,y_2) \in f$, then $y_1 = y_2$. This means that each element of X is related to a unique element of Y. y is called the map or image of x under the mapping f and is written as $y = f(x)$.

Example

1. A single real valued function f(x) of a single real variable x is identical with the mapping $f : R \rightarrow R$ where R is the set of all real numbers.

When $y = f(x)$ and $x, y \in R$, we say that y is a real single-valued function of the real variable x iff for each $x \in R$, there corresponds a unique real number y, i.e., f(x) which we calculate from the formula $y = f(x)$. Hence the function 'f' associates with each $x \in R$, one unique $y \in R$. Thus we note that (i) the domain of the function f is R and (ii) for each $x \in R$, there is exactly one ordered pair (x,y) determined by f. These are exactly the characteristics

of a mapping $f: R \to R$. Hence the real single valued function $f(x)$ is identical with the mapping $f: R \to R$.

2. Let R be the set of all real numbers and R^+ be the set of all non-negative real numbers. Show that the mapping $f: R \to R^+$ defined by $f(x) = x^2$ for each $x \in R$ is an onto mapping.

For, since the square of every real number $x \in R$ is non-negative real number, it will belong to R^+. Also every element in R^+ will surely occur as the second element of some ordered pair in f because given $a \in R^+$, it is always possible to find some $b \in R$ such that $b^2 = a$. Consequently, the mapping $f: R \to R^+$ is onto mapping.

3. Let R_1 be the set of all real numbers greater than -1. Then show that the mapping $f: R \to R_1$ defined by $f(x) = x^2$ for each $x \in R$ is into mapping.

Since the range of mapping is R^+ which is a proper subset of R_1, hence the range of f is not the set R_1 and therefore, the mapping is into.

Note that the mapping $f: R \to R$ defined by $f(x) = x^2$ is also an into mapping. It is said that 'f is a mapping of R into itself'.

4. The mapping $f: R \to R_1$ defined by $f(x) = x^2$ of Ex. 3 is an example of a mapping which is many-to-one and into; while $f: R \to R^+$ defined by $f(x) = x^2$ is an example of a mapping which is many-to-one and onto.

5. The mapping $f: R^+ \to R$ defined by $f(x)=x^2$ is an example of one-to-one mapping which is into. It is one-to-one because no two distinct elements of R^+ can be mapped with the same element of R. It is into because the range of f is a proper subset of R.

6. The mapping $f: R^+ \to R^+$ defined by $f(x) = x^2$ is an example of one-to-one mapping which is onto. It is one-to-one because no two distinct elements of R^+ can give rise to the same element of R^+ and it is onto because the range of f is the whole set R^+.

7. The mapping $f: R \to R$ defined by $f(x) = \cos x \ \forall x \ R$ is neither one-to-one nor onto, for

(i) $f(x) = f(2n\pi + x)$ where n is an integer showing many elements of the domain give rise to the same element in the range. Hence mapping is not one-to-one.

(ii) $f(x)$ always lies in the closed interval $[-1, 1]$ and hence the mapping is not onto.

Now if we vary the domain and range of the above mapping according to the rule $f(x) = \cos x \ \forall \ x$ in the domain in a manner that

(a) $f: [0, \pi] \to R$, it will be one-to-one mapping but not onto.

(b) $f: R \to [-1, 1]$, it will be an onto mapping but not one-to-one.

(c) $f: [0, \pi] \to [-1, 1]$, it will be one-to-one as well as onto.

Composition of Functions

Two functions f and g are said to be composable iff the range of f is a subset of the domain of g. Thus if $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are two functions, then since the range of f is a subset of the domain of g, i.e. Y, these two functions are composable. Let us define a function $\phi : X \rightarrow Z$ by associating with each element $x \in X$, the element $\phi(x)$ of Z such that $\phi(x) = g[f(x)]$. This function ϕ is also denoted by $g \circ f$ and is read as 'g composition f'. Thus $g \circ f = g\{f(x)\}$. These mappings can be shown in a pictorial form as follows.

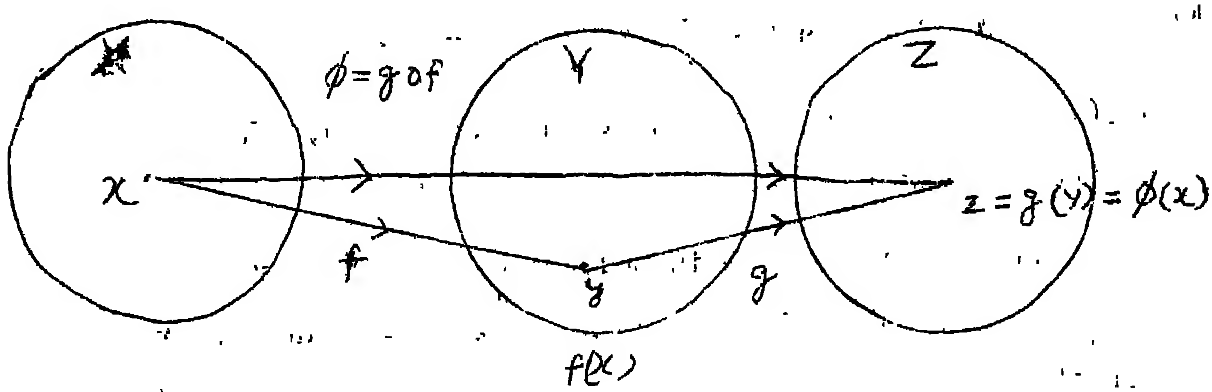


Fig.1.11

If there are two mappings $f: R \rightarrow R$ and $g: R \rightarrow R$, then since range of f will always be a subset of the domain of g, f and g are composable (i.e., $g \circ f$ exists). Also, since the range of g will always be a subset of the domain of f, the

mappings, g and f are composable (i.e., f o g exists). Thus here g o f and f o g both are defined. It will be found that, in general, g o f $\neq$ f o g . For instance, let f: $R \rightarrow R$ be a mapping defined by $f(x) = x^2$ for every $x \in R$ and g: $R \rightarrow R$ be a mapping defined by $g(x) = \cos x \ \forall x \in R$.

Then $(g \circ f)(x) = g\{f(x)\} = g(x^2) = \cos x^2$
$\forall \ x \in R$,
and $(f \circ g)(x) = f\{g(x)\} = f(\cos x) = (\cos x)^2$
$\forall \ x \in R$ $\therefore$ Obviously $g \circ f \neq f \circ g$.

Thus the composition of two mappings is not necessarily commutative even if g o f and f o g both are defined,

Example

f_1, f_2 f_3, f_4, f_5 and f_6 are functions from the rationals in the open interval (0,1) to the rationals defined by
$f_1 : x \rightarrow x$, $f_2 : x \rightarrow 1-x$, $f_3 : x \rightarrow \frac{1}{x}$, $f_4 : x \rightarrow \frac{1}{1-x}$,
$f_5 : x \rightarrow \frac{x}{x-1}$, $f_6 : x \rightarrow \frac{x-1}{x}$. Using composition of the mappings as the operation, an operational table may be constructed as follows.

o	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2						
f_3						
f_4			f_5			
f_5						
f_6					f_3	

Notice that in order to determine that $f_6 \circ f_5 = f_3$, the students have to be able to simplify as follows:

$$(f_6 \circ f_5)(x) = f_6\{f_5(x)\} = f_6\left(\frac{x}{x-1}\right)$$

$$= \frac{\frac{x}{x-1} - 1}{\frac{x}{x-1}} = \frac{1}{x} = f_3(x)$$

$\therefore f_6 \circ f_5 = f_3$

Similarly $(f_4 \circ f_3)(x) = f_4\{f_3(x)\} = f_4\left(\frac{1}{x}\right)$

$$\frac{1}{1-\frac{1}{x}} = \frac{x}{x-1} = f_5(x)$$

$\therefore f_4 \circ f_3 = f_5$.

By carefully observing the operational table, the students will be able to discover that the set $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ under this composition is non-commutative.

<u>Inverse of an Element Under a Mapping</u>

If mapping $f: A \rightarrow B$, then the inverse element of an element $b \in B$ is $a \in A$ if $f(a) = b$ and we write $f^{-1}(b) = a$. It is likely that a particular element of set B may not have or may have more than one inverse element. This depends upon the type of mapping. The set of inverse elements of $b \in B$ is denoted by

$$f^{-1}(b) = \{a:\ a \in A,\ f(a) = b\}.$$

Inverse Function

If $f: A \to B$ is an one-to-one function, then we define a function g from B to A, i.e. $g: B \to A$ such that $g(b) = f^{-1}(b)$. This function g is called the inverse function of f and is written as f^{-1}.

Example

If $A = \{a_1, a_2, a_3\}$ $B = \{b_1, b_2, b_3\}$ and $f: A \to B$, $f(a_1) = b_2$, $f(a_2) = b_1$, $f(a_3) = b_3$. Then $f^{-1}: B \to A$ and $f^{-1}(b_1) = a_2$, $f^{-1}(b_2) = a_1$, $f^{-1}(b_3) = a_3$.

1.4 Binary Operations

Let A be a non-empty set. Any function or map $f: A \times A \to A$ is called binary operation on A or binary composition in A.

The image of an ordered pair $(a,b) \in A \times A$ under f is denoted by a f b. Mostly addition ' + ', multiplication ' x ' or '.' are used as binary operations.

If 'O' is a binary operation on A then we also say that A is closed with respect to the binary operation O. 'O' is being operation on A iff $(a.b) \in A$.

Example

1. Addition is a binary operation on the set N of natural numbers, where, as subtraction is not a binary operation on N.

For addition of two natural numbers is always a natural number, but subtraction of two natural numbers is not necessarily a natural number.

2. Addition is a binary operation on the set R of real numbers. For sum of two real numbers is always a real number.

3. Subtraction is binary operation on R. For difference of two real numbers is a real number.

4. Division is not a binary operation on R but division is binary operation on R $-\{0\}$.

5. Multiplication is a binary operation on the set $A = \{1, -1, i, -i\}$ of four fourth roots of unity. For product of any two elements of A is an element of A, which can be seen in the following composition table.

	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

6. Addition, subtraction and multiplication are binary operatio on the set of matrices. For, let A and B be two matrices of the same type m x n. Then their sum (to be denoted by A+B) is defined to be the matrix of the type m x n obtained by adding the corresponding elements of A and B. Thus if

$$A = [a_{ij}]_{m \times n} \quad \text{and} \quad B = [b_{ij}]_{m \times n}$$

then $$A + B = [a_{ij} + b_{ij}]_{m \times n}$$

matrix of order $m \times n$.

Similarly, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ are two matrices such that the number of columns in A is equal to the number of rows in B. Then the $m \times p$ matrix $C = [c_{ik}]_{m \times p}$ such that $C_{ik} = \sum_{j=1}^{n} a_{ij} \; b_{jk}$ is multiplication of the matrices A and B in that order and we write $C = A\,B$

7. Multiplication is a binary operation over the set of permutation. For, let $X = \{ a_1, a_2, \ldots, a_n \}$. Let $f : X \to X$ and $g : X \to X$ be one-one onto maps. Then f, g are permutation of degree n. Clearly $g \circ f : X \to X$ and $f \circ g : X \to X$ are one-to-one maps. Hence $f \circ g$ and $g \circ f$ are permutations of degree n. Thus the product of two permutations of degree n is a permutation of the same degree.

8. Addition is a binary operation on the set of polynomials. For addition of two polynomials is a polynomial.

<u>Commutatativity and Associativity of Binary Operations.</u>

A binary operation 'O' over a set A is said to be commutative if two elements a, b of A, $a \circ b = b \circ a$.

<u>Example</u>

1. The operation '+' is commutative in N, I, Q, R and C.

2. '+' is commutative over the set of matrices i.e., $A + B = B + A$ for any two matrices A and B of order $m \times n$.

3. '·' is not commutative over the set of matrices.

Associativity:

A binary operation 'O' over a set A is said to be associative if for every three elements a, b, c of A, (a o b) oc = a o (b o c).

Example.

1. Ordinary addition is associative over the set N, I, Q, R and C.

2. Ordinary subtraction is neither commutative nor associative over the sets I, Q, R, C.

3. Addition of matrices is associative. i.e. if A, B, C are three matrices each of the type m x n, then (A + B) + C = A + (B + C).

4. Matrix multiplication is associative; if conformability is assured i.e.,

(A B).C = A (B C) if A, B, C are m x n , n x p , p x q matrices respectively.

Identity Element with Respect to a Binary Operation.

Suppose 'O' is a binary operation defined over the set A. Then the set A is said to have an identity element with respect to the binary operation 'O' if ∃ e ∈ A such that when e is applied to an element a ∈ A under the given operation O, leaves the element unchanged i.e.,

$$e \circ a = a \circ e = a \qquad (1)$$

When $e \circ a = a$, we say that e is a left identity for 'o'.
When $a \circ e = a$ we say that e is a right identity for 'O'.
When (1) holds, we say that e is a (two sided)identity for O.

Example.

1. The set N of natural numbers has no identity element with respect to addition, but it has the identity element 1, with respect to multiplication.

2. The sets I, Q, R, C each has an identity with respect to addition as well as multiplication, these being 0 and 1 respectively.

3. Null matrix or zero matrix is additive identity in the set of matrices.

$$(A + O = O + A = A)$$

4. Unit matrix is multiplicative identity in the set of matrices.

$$(A I = A = I A)$$

Inverse of an Element of A

Let 'O' be the binary operation defined on the set A. Then an element b of A is said to be inverse of an element a of A with respect to the identity e. (relative to the binary operation O) iff

$$a \circ b = b \circ a = e \qquad (1)$$

b is called left inverse of a relative to e if $b \circ a = e$, while b is called the right inverse of a relative to e if $a \circ b = e$, while if (1) holds b is called the inverse of a. On the basis of the definition of an inverse of an element it is easily seen that a is inverse element of b.

Example.

1. The set N has no inverse for any of its elements with respect to additions as well as multiplication.

2. The set I has inverse for each of its elements with respect to addition but does not have inverse for all elements with respect to multiplication. Find which elements of I have inverses with respect to multiplication.

3. The set Q (R, C) has inverse for each of its elements with respect to addition and for each of its elements except 0 with respect to multiplication.

4. In case of matrix, the matrix $-A$ is the additive inverse of the matrix A. Obviously $-A + A = 0 = A + (-A)$.

3. Learning Outcomes.

(a) Essential Learning Outcomes for All.

After going through this chapter a student should be able to

(i) Understand the concept of a set, relation and function.

(ii) perform the operations related to the sets such as union, intersection, complementation and difference.

(iii) have the knowledge of different types of relations, to identify and distinguish between them.

(iv) have the knowledge of the function and its different types and have ability to distinguish between them. Students should know the meaning and understand the inverse of a function.

(v) know the difference between relation and function and difference between their domains. Here it may be noted that the domain of a relation from A to B is a subset of A and its range is a subset of B where as domain of a function is the entire set A.

(vi) able to perform the composition of functions.

(vii) Understand the meaning of a binary operation. Given a binary operation the pupil should be able to state whether the operation is commutative, associative admitting identity and inverses.

(b) <u>Learning Outcomes for the Higher Group.</u>

After going through this chapter, students of higher group should be able to

(i) have ability to use the knowledge gained in solving problems connected with sets, relations and functions.

(ii) Solve difficult and challenging problems.

(iii) Use their knowledge for solving puzzles, quiz etc.

(iv) Construct composition table with the help of a given finite set and binary operation.

4. Teaching Strategies.

Learning takes place effectively if the subject is taught in an interesting way and it is ensured that the students are understanding whatever is being taught. For this the content should be related to day to day life situation. It should be from 'easy to difficult' and 'from concrete to abstract'. Below are given some hints for effective teaching of this chapter. However it depends largely on a teacher's resourcefulness that how effective his teaching is.

The students have already learnt the set theory in the previous classes, therefore the teacher should teach sets mostly as a recall. The relation, function and binary operations should be taught in detail.

Motivation.

Ordered Pair.

The ordered pair should be explained by introducing pairs as representing a certain fact. For example if pair (a,b) represents that a and b live in the same country, then (Ashok, Arif) will mean Ashok and Arif live in the same country. Similarly if (x,y) means x is father of y, then (Rakesh, Vijay) will mean

Rakesh is father of Vijay. Now in the first case if we write (Arif and Ashok live in the same country which expresses the same fact as expressed by (Ashok, Arif). But in the second case if we change the order we get (Vijay, Rakesh) which means Vijay is father of Rakesh which is not true. Thus if the pair (a,b) represents the fact that 'a is father of b' then we can not change the order. The order is important; such pairs in which the order of elements is important, are called ordered pairs. Similarly

(a,b) ≡ a and b are class fellows is not an ordered pair.

(a,b) ≡ a is capital of b is ordered pair.

1.2 Relations

The concept of relation should be introduced by taking some prevalent relation in our society for example the relation of 'is maternal uncle of '. It should be made clear to the students by asking the following questions:

(i) When a person x is maternal uncle to another person y ?

(ii) If x is the brother of y's mother, who will be matern uncle 'of whose'

The answer to first question illustrates that a relation is established when a particular condition is fulfilled. The answer to second querry shows the importance of order.

In order to show that a relation is from a set A to another set B (B may be equal to A), the following two sets of adults and B of children should be taken. The arrows show

which element of A is maternal uncle of which element of B.

A = { Mohan, Joseph, Karim, Anand }

B = { Pintoo, Simon, Latif, Ashok, Ram }

This relation R is represented by the set of ordered pairs
(Mohan, Pintoo), (Joseph ,Simon), (Karim, Latif), (Mohan, Ashok
which is a subset of A x B.

Another example of a relation may be taken as follows:

Consder two sets of persons X and Y. Suppose X consists of persons:

a, b, c, d, e

with ages 5, 10, 14, 17, 20

years respectively and Y consists of the persons,

p, q, r, s, t

with ages 10, 25, 50, 28, 34

years respectively. Thus

$$X = \{ a, b, c, d, e \}$$
$$Y = \{ p, q, r, s, t \}.$$

Now observing the ages of the persons, the relation between the ages of a and p, c and s, and d and t is in the ratio 1 : 2 . If we assume that a person εX will be related a person εY if the ratio of their ages is 1: 2. Thus as

regards their ages, each of the pairs (a, b), (c, s), (d,t) is in the said relation. If we denote this relation by a single letter R, then this relation will be denoted by

a R p, c R s and d R t .

The relation R, therefore , is a set of ordered pairs

(a,p), (c, s), (d,t) i.e.

$$R = \{ (a,p), (c,s), (d,t) \}$$

which is a subset of X x Y

Each pair is said to be an ordered pair since each pair has been written in one and the same order. There are pairs in the above example, which are not in the above relation R. For instance (a,q) $\notin$ R, since age of a: age of q = 1:5 , which is a relation different from the above relation R. This fact is also **written** as a $\not R$ q.

As a next example, suppose we collect pairs of persons one from X and the other from Y whose ages are in the relation 1: 5. Obviously there are only two pairs viz (a,q) and (b,r) which are in this new relation say s . In this case the relation s contains only two elements.

These examples show that a relation between two sets X and Y is nothing but a set of ordered pairs (x,y) with x $\in$ X and y $\in$ Y. The number of elements i.e. ordered pairs in each relation between two sets X and Y will depend on the nature of the relation prescribed.

It may happen that a relation between X and Y may contain

no elements at all. For instance, in the above example, if the ages of two persons - one in X and the other in Y are to be in the ratio 1 : 3 , there is no pair which is this relation. Such a relation is called an empty relati which is the smallest relation.

There is another extreme case in which a relation ma contain all the ordered pairs (x,y) with $x \in X$ and $y \in Y$. For instance suppose persons in the set X are right-hand badminton players, and those in the set Y are left-hande badminton players and we have to collect pairs based on relation that in each pair one player is to be right-han and the other left-handed. If we denote this relation by then

$$t = X \times Y$$

Obviously this relation is the largest possible relatio between X and Y.

These examples show that any relation R between two X and Y is always such that $R \subset X \times Y$.

Since a relation is a collection of ordered pairs r than ordered tripples etc. We call it a binary relation But it is customary to use the word 'relation' in place 'binary relation'.

Equivalence Relation and Partition.

This topic may be motivated to the students in the way.

1. Let A represent the set of people residing in a town. Now we define a relation on A namely a R b iff a and b live in the same house, assuming that all people live in one or the other house. Then this relation is an equivalence relation. This will partition the set A. The set of houses forms partition of A. Thus if P is the partition of A, then

$$P = \{ A_i : A_i \text{ is the set of people living in the house } i \}$$

2. Consider a school which has classes 1 to 12. Let S denote the set of students in the school. Let us define a relation R on S as follows:

a R b iff a and b are in the same class. This relation is an equivalence relation which partitions the set S into 12 disjoint Sets. Each class forms one such disjoint subset, called equivalence class. In this case

$$P = \{ S_i : S_i \text{ is the set of students in class } i,\ i \text{ varies from 1 to 12.} \}$$

More precisely

S_1 = set of students in class 1,

....

S_{12} = set of students in class 12.

Thus

$$P = \{ S_1, S_2, \ldots, S_{12} \}$$

3. The following relation in the set of all scooters at a particular scooter stand should be taken.

x R y if scooter x is of the same make as y. This example will help students to see the concept of 'partitioning'. A particular equivalence class will contain the scooter of the same make.

1.3 Functions

Functions should be introduced as particular cases of relations, i.e., a relation from A to B whose domain is the whole Set A and every element of A has no more than one relation in B. A relation from a set of children to the set of all male adults, defined as 'is son of' is an example of a function.

Injective, surjective and bijective functions can be demonstrated by taking the set of students in the class and the of chairs. The students should be made to sit on the chairs so that mapping of different types may be defined.

The composition of mapping should be explained through charts and diagrams.

Misconceptions/Common Errors.

Experiences show that the students have the following type of learning difficulties and misconceptions.

(i) The students can not distinguish between the singleton set and the null set. They take the singleton $\{0\}$, $\{\emptyset\}$

as null set . It should be explained to them that

$\{0\}$ has one element namely 0,

$\{\emptyset\}$ has one element $\emptyset$.

while $\emptyset$ or $\{\ \}$ has no element

(ii) They confuse among the symbols

$\in$, $\subset$ & $\subseteq$.

The difference should be explained.

(iii) They can not see the difference between a relation and a function, this should be explained (see the content analysis).

(iv) The students some times misconceive that as in the case of a function from A to B the domain is the set A, so is the case for a relation from A to B. This should be cleared that in case of a relation from A to B the domain of R = $\{a : (a,b) \in R\}$ which may or may not be equal to A.

(v) Often g o f and f o g are not understood properly. The students take g o f for f o g and vise versa. The meaning of these should be explained clearly.

Solutions/hints for difficult Problems of Text Book.

Exercise 1.1.

1. Proof: Let $x \in A^c \setminus B^c$

This $\Rightarrow x \in A^c$ and $x \notin B^c$

$\Rightarrow x \notin A$ and $x \in B$

$\Rightarrow x \in B$ and $x \notin A$

$\Rightarrow x \in B \setminus A$

$\therefore \quad A^c \setminus B^c \subset B \setminus A$ (i)

Again

$$x \in B \setminus A \Rightarrow x \in B \text{ and } x \notin A$$
$$\Rightarrow x \in A^c \text{ and } x \notin B^c$$
$$\Rightarrow x \in A^c \setminus B^c$$

$\therefore \quad B \setminus A \subset A^c \setminus B^c$ (ii)

From (i) and (ii)

$$A^c \setminus B^c = B \setminus A$$

4. Let (a,b) , (c,d) and $(e,f) \in N \times N$

(i) R is reflexive

We know that addition of natural numbers satisfies commutative law viz

$$a + b = b + a \quad \forall \; a, b \in N$$

Hence (a,b) R (a,b). Therefore R is reflexive.

(ii) R is symmetric

Let (a,b) R (c,d)

Now $(a,b)\ R\ (c,d) \Leftrightarrow a + d = b + c$ (by def. of R)

$\Leftrightarrow b + c = a + d$ ($\because$ equality is symmetric)

$\Leftrightarrow c + b = d + a$ ($\because$ '+' is commutative)

$\Leftrightarrow (c,d)\ R\ (a,b)$

$\therefore$ R is symmetric

(iii) R is transitive

Let $(a,b)\ R\ (c,d)$, $(c,d)\ R\ (e,f)$

Now $(a,b)\ R\ (c,d) \Leftrightarrow a + d = b + c$ (by def. of R)

$(c,d)\ R(e,f) \Leftrightarrow c + f = d + e$ (by def. of R).

$\therefore$ (a,b) R (c,d) and (c,d) R (e,f)

$\Leftrightarrow a + d + c + f = b + c + d + e$

$\Leftrightarrow a + f + (c + d) = b + e + (c + d)$

$\Leftrightarrow a + f = b + e$

$\Leftrightarrow$ (a,b) R (e,f)

Hence R is transitive

Thus R is an equivalence relation

6. Suppose $x_1, x_2 \in A$ and $x_1 \neq x_2$

$(g \circ f)(x_1) = g(f(x_1))$ (definition of composition of functions).

$(g \circ f)(x_2) = g(f(x_2))$ (definition of composition of functions)

Since f , g are one-one

$\therefore x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

and $y_1 \neq y_2 \Rightarrow g(y_1) \neq g(y_2)$

Now $f(x_1) \neq f(x_2) \Rightarrow g(f(x_1)) \neq g(f(x_2))$.

i.e. $(g \circ f)(x_1) \neq (g \circ f)(x_2)$ if $x_1 \neq x_2$

Therefore g o f is also one-one by the definition of 1-1 functions.

9. Yes, since $a \circ b = b \circ a \;\forall\; a,b,c \in \{a,b,c\}$

11. Proof.

There are $^nC_1$ subsets each containing one single element

There are $^nC_2$ subsets each containing two elements

....

....

There are ^nC_n subsets i.e. one subset containing n elements.

Lastly $\emptyset$ is also one subset of A. Therefore the total number of subsets of

$$A = 1 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$$

12. (i) $f : R^+ \rightarrow R$ s.t. $f(x) = x^2$ is one-to-one and not onto

(ii) $f : R \rightarrow [-1, 1]$ s.t. $f(x) = \cos x$ is not one to one but onto

(iii) $f: R \rightarrow R$ defined by $f(x) = \cos x \ \forall x \in R$ is neither one-to-one nor onto. For (i) $f(x) = f(2n\pi + x)$ i.e. different elements have same images.

(iv) $f(x)$ always lies in the closed interval $[-1, 1]$ and hence not onto

13. To prove that f and g are bijections, we shall prove that f is one-one. Let

$x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$

Then $f(x_1) = f(x_2) \Rightarrow g[f(x_1)] = g[f(x_2)]$

$$\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$$

$$\Rightarrow I_A(x_1) = I_A(x_2) \quad [\because g \circ f = f \circ g = I_A]$$

$$\Rightarrow x_1 = x_2$$

Hence f is one-one.

Similarly, to prove that g is one-one, we see that $g(x_1) = g(x_2)$

$$\Rightarrow f(g(x_1)) = f(g(x_2))$$
$$\Rightarrow (f \circ g)(x_1) = (f \circ g)(x_2)$$
$$\Rightarrow I_A(x_1) = I_A(x_2)$$
$$\Rightarrow x_1 = x_2$$

Hence, g is one-one

Now, to prove that f is onto, let $x, y \in A$ and $g(y) \in A$. Let $g(y) = x$.

Then, $g(y) = x \Rightarrow f[g,(y)] = f(x)$

$$\Rightarrow (f \circ g)(y) = f(x)$$
$$\Rightarrow I_A(y) = f(x)$$
$$\Rightarrow y = f(x)$$

Thus $y \in A \Rightarrow \exists x \in A$ such that $f(x) = y$.

Similarly g is onto. Thus f and g are bijections. Hence f^{-1} and g^{-1} exist.

Now, to prove that $f^{-1} = g$, we see that

$$f \circ g = I_A \Rightarrow f^{-1} \circ (f \circ g) = f^{-1} \circ I_A = f^{-1}$$
$$\Rightarrow (f^{-1} \circ f) \circ g = f^{-1}$$
$$\Rightarrow I_A \circ g = f^{-1}$$
$$\Rightarrow g = f^{-1}$$

Similarly, it can be proved that $f = g^{-1}$.

15. Let (a,b), (b,c) and (e,f) N x N

(i) <u>R is reflexive</u>

Since multiplication of natural numb.r is commutative.

$\therefore$ a.b = b. a

This $\Rightarrow$(a,b) R(a,b)

$\therefore$ R is reflexive.

(ii) <u>R is symmetric</u>

Let (a,b) R (c,d)

Now (a,b) R (c,d) $\Rightarrow$ a.d = b.c. (by def. of R)

$\Rightarrow$ b.c. = a.d ($\because$ equality is symmetric)

$\Rightarrow$ c.b = d.a. (commutative law holds in N)

$\Rightarrow$ (c,d) R (a,b)

$\therefore$ R is symmetric

(iii) Let (a,b) R(c,d) and (c,d) R (e,f)

(a,b) R(c,d) $\Rightarrow$ a.d. = b.c.

and (c,d) R (e,f) $\Rightarrow$ c.f = d.e.

$\therefore$ (a,b) R(c,d) and (c,d) R (e,f)

$\Rightarrow$ a.d. c.f. = b.c. d.e.

$\Rightarrow$ a.f. c.d. = b.e. c.d

$\Rightarrow$ a.f. = b.e

$\Rightarrow$ (a,b) R (e,f)

$\therefore$ R is transitive

Hence R is an equivalence relation.

16. $f(f(x)) = f(x^2 - 3x + 2)$

$= (x^2 - 3x + 2)^2 - 3.(x^2 - 3x + 2) + 2$

$= x^4 - 6x^3 + 10x^2 - 3x$

17. (i) $(f \circ g)(x) = f(g(x)) = f(2x-3)$

$$= (2x-3)^2 + 3(2x-3) + 1$$
$$= 4x^2 - 6x + 1$$

$$(g \circ f)(x) = g(f(x)) = g(x^2 + 3x + 1)$$
$$= 2(x^2 + 3x + 1) - 3$$
$$= 2x^2 + 6x - 1.$$

[Note: $f \circ g \neq g \circ f$]

Similarly $f \circ f$ and $g \circ g$ can also be found out.

19. Since $A \subset B$ need not imply $B \subset A$. Therefore relation of inclusion is not symmetric. Hence it is not an equivalence relation.

Test Paper

1. Prove that a relation R defined in a set A is symmetric iff $R = R^{-1}$

2. Let R be a relation in the set I of integers such that $x R y \Leftrightarrow x-y$ is an even integer. Show that R is an equivalence relation.

3. Classify the following functions as surjections, Injections or bijection. Also determine their ranges;

   $f : C \to R \quad : f(z) = |z|$

   Ans: No injection.

   and No surjection.

4. Is the function $f : N \to N$ defined by $f(x) = 2x + 3$ surjective? Ans: No

5. Let a function $f: R \to R$ be defined as follows:
$f(x) = x^3 + 3$
Prove that f is one-one and onto in every case. Obtain a formulae that defines f^{-1}. Ans. $f^{-1}(x) = (x-3)^{1/3}$

6. If the function f and g are both bijection prove that their composite $g \circ f$ is also a bijection.

7. Given $A \times B = \{(1,1), (1,2), (2,1), (2,3), (2,2), (1,3)\}$
Find A and B   Ans: $A = \{1,2\}$
$B = \{1,2,3\}$

8. $A = \{1,2,3\}$, $B = \{1,2,3,4\}$
Define R from A to B such that a R b if $b = a+1$. Write R as a set of ordered pairs.   Ans: 1 R2, 2R3, 3R4
& $\{(1,2), (2,3), (3,4)\}$

9. Prove that $A^c \cap (B^c \cup C^c) = [A \cup (B \cap C)]^c$

<u>Additional Reading Materials.</u>

1. Some Introductory Lessons on Modern Mathematical Concepts, P.L.Bhatnagar, East West Press.

2. Modern Mathematics, Vol. I & II, Papy, Mac Millan.

3. Modern Mathematics (Course I, Course 11, and Course 111) John Sweeney, Gill and Mac Millan.

4. A Survey on Modern Mathematics, Birkhof and Maclene.

5. Modern Algebra, I.N. Herstein, Vikas Publisning House, Pvt. Ltd.

6. What is Mathematics, R.Cowrant & H.Robbins, Oxford University Press.

7. Theory and Problems of Group Theory. B. Baumslag and B. Chandler Schaum's outline series, Mc Graw Hill Book Company.

8. Introduction to Modern Algebra Neal H. Mc Coy, Bostons Allyn and Bacon Inc.

CHAPTER-2

COMPLEX NUMBERS

1. Introduction

The need for complex numbers arose when mathematicians tried to solve the equations of the form $x^2 + k = 0$ where k is a positive real number; for example, $x^2 + 1 = 0$.

Its solution involves the square root of a negative real number. Today it is well known that complex numbers play a very important role in the field of science and technology. The students will be using complex numbers in many branches of mathematics like Trigonometry, algebra, analysis, etc. and also in physics and a number of other fields. This makes the study of complex numbers more important.

Let us now consider the equation $x^2 = -1$ i.e. $x^2+1 = 0$. There is no real solution of this equation because the square of any real number can not be negative. This difficulty was first stated by the famous Hindu Mathematician Mahavira at about 850 A.D. He wrote "In the nature of things, a negative is not a square. It has therefore no square root". The Italian Mathematician Luca Pacioli explained in his article 'Summa Arithmetica' in 1494 that the quadratic equation $x^2 + c = bx$ could be solved only if $b^2 > 4c$. This meant that the solution of an equation such as $x^2 = -a^2$ where a is a real number, was impossible in the set of real numbers.

The origin of the set of complex numbers is credited to the Italian Jerome Cardano. In 1545 A.D. in his 'Aro Magna', Cardano used the square root of a negative number in the solution of a problem such as "divide 10 into two parts whose product is 40". This amounts to solving the equation $x(10-x) = 40$. The square roots

of the negative numbers were called 'sophistic' numbers by Cardano.

The square roots of negative numbers were considered as fictitious and hence, Descartes gave them the name "Imaginary numbers" in 1630 A.D.

In 1572 A.D., the Italian mathematician Rafael Bombelli, motivated by the solution of cubic equations, also stated the rules of operations with square roots of negative numbers. The letter 'i' to designate $\sqrt{-1}$ was first used in 1748 A.D. by the Swiss mathematician Leonhard Euler.

A geometrical interpretation of a complex number was given by Wassel in 1797 A.D., and by Argand in 1806 A.D. The German mathematician Carl Friedrich Gauss in 1832 A.D. named a number having form "$a+b\sqrt{-1}$" as 'a complex number' to distinguish it from numbers of the form $a\sqrt{-1}$.

As a consequence of a theorem proved by Gauss in 1797 (Fundamental Theorem of Algebra), the complex number system is adequate for the purpose of solving polynomial equations of all types. Finally in 1837 A.D., Sir William R. Hamilton published his rigorous development of the set of complex numbers as the set of ordered pairs of real numbers.

Now the theory of complex numbers has become a highly developed branch of mathematics. The students will often be required to use the complex numbers while studying the other branches of mathematics. Hence the teaching of complex numbers becomes very important and every attempt should be made to teach it effectively. Though every aspect of the content in this chapter is very important, yet concepts of a complex number, Argand Diagram, Laplace-de-Moivres' formula etc. deserve special attention.

2. Content Analysis:

In this section, the number of each subsection is in accordance with that of the text book.

The complex number system is an extension of the real number system in that the complex numbers are defined in terms of the real numbers and every real number x can be viewed as a complex number of the form x+io. Thus the set of real numbers can be put into one-one correspondence with a certain subset of the set of complex numbers. It may be noted that this subset has as its members the complex numbers of the form a+io.

2.1 The Algebra of Complex Numbers:

Complex numbers can be viewed in different ways as follows:-

(i) Numbers of the form a+ib where $a, b \in R$

(ii) As an ordered pair of real numbers.

It is better to introduce a complex number as an ordered pair of real numbers such as (4,3), $(\sqrt{2},\pi)$, $(\frac{1}{3}, o)$ etc. so that the ordered pair (x,y) is actually

$(x,y) \equiv x+iy$, $x, y \in R$ and $i = \sqrt{-1}$. In the complex numbers (x,y), i.e. x+iy, x is called the real part and y the imaginary part.

In the text book, purely imaginary numbers have not been discussed. It should be made clear to the students that a complex number (x,y) is purely imaginary if x = 0, i.e. those numbers are purely imaginary which are the square roots of negative real numbers.

The equality of two complex numbers also has not been defined in the text book. It is very important to define this concept as it lead to important deductions. It may be defined as follows:

Two complex numbers $z = x+iy$ and $z' = x' + iy'$ are equal iff $x = x'$ and $y = y'$, i.e. $z = z' \Leftrightarrow x = x'$ and $y = y'$. Thus two complex numbers are equal iff their real and imaginary parts are separately equal.

The concept of equality of two complex numbers leads to the important result that $a+ib = 0 \Rightarrow a = 0 \wedge b = 0$.

The operations of addition and multiplication of two complex numbers have been defined and discussed precisely in the text book. The properties of these operations, i.e. associative, commutative etc. have been proved. Identity element for each operation and inver of a complex number have been determined.

While explaining to the students the identity element of addition, the following points should be made clear:

(i) 0 means $0 + io$

(ii) 0 is the identity element of addition, and it is unique.

Similarly the identity '1' for multiplication means $1+io$ and it is unique.

In order to prove the uniqueness of the identity element for an operation, the following procedure may be adopted.

Let, if possible, 0 and $0'$ be the identity element for additio Now if $0, 0' \in C$ and $z \in C$, then

$$z + 0 = z \qquad (1)$$

$$z + 0' = z \qquad (2)$$

$$\therefore \quad 0 = 0' \text{ because } 0' + 0 = 0'+0.$$

Similarly it should be emphasized and proved that inverse of an element (complex number) with respect to an operation is unique. It may be proved in the following way:-

Let, if possible, z_1 and z_2 be two additive inverses of z

while $z, z_1, z_2 \in C$.

Therefore $z + z_1 = 0$

and $z + z_2 = 0$

Hence $z + z_1 = z + z_2$

Or, $z_1 = z_2$

In this section (2.1) while discussing the properties of addition and multiplication, it has been pointed out that set C with these operations is a field. It will be clear if it is told to the students what a 'field' means. A field may be defined as follows:-

If A is a nonempty set on which there are defined two binary operations (called here + and ') and if for any arbitrary elements $a, b, c \in A$,

(i) $a + b = b+a$

(ii) $(a+b)+c = a + (b+c)$

(iii) an element $0 \in A$ such that $a+0 = a$

(iv) if $a \in A$, there exists $x \in A$ such that $a+x = 0$

(v) $(a.b).c = a.(b.c)$

(vi) $a.(b+c) = a.b + a.c$,

then we say that A is a <u>ring</u>.

In addition to above properties, if (vii) $a.b = b.a$

(viii) an element $e \in A$ such that $ae = ea = a$

(ix) for each $a \in A$ $\exists$ $y \in A$, such that $a.y = y.a = e$

hold, then A is said to be a field.

The important concept of conjugate of a complex number has been given and the following properties related to it are proved in the text book:

(i) $\overline{(z_1 \pm z_2)} = \bar{z}_1 \pm \bar{z}_2$

(ii) $\overline{z_1 z_2} = \overline{z_1 \cdot z_2}$

(iii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ where $z_2 \neq 0$

Modulus or absolute value of a complex number has been also defined and the following properties pertaining to it have been proved:-

i) $|z_1 + z_2| \quad |z_1| + |z_2|$

ii) $|z_1 - z_2| \quad \big||z_1| - |z_2|\big|$

iii) $|z_1 z_2| = |z_1| \cdot |z_2|$

iv) $\left|\frac{z_1}{z_2}\right| \quad \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$

2.2 <u>Argend Diagram:</u>

The representation of complex numbers as points on a plane is known as <u>Argand diagram</u>. This has been discussed in this section, Complex plane, real axis and imaginary axis have been defined.

Some of the operations and properties discussed in Art. 2.1 have been illustrated through Argand diagram in this section. It will be better if some more properties are also illustrated through this method. For example, to prove associative property of addition, we may draw the following diagram:

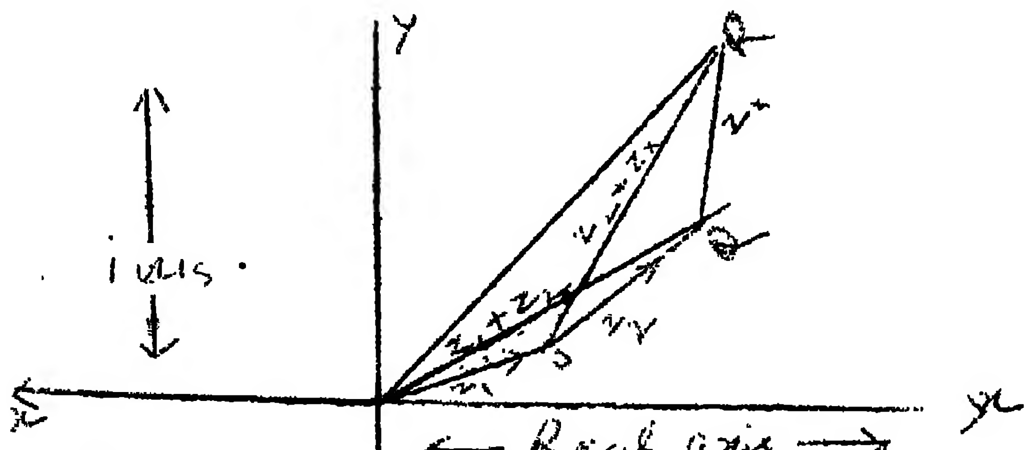


Fig. 2.1

$OQ = z_1 + z_2$

$OR = OQ + OR$

$= (z_1 + z_2) + z_3$

$= z_2 + z_3$

$OR = OP + PR = z_1 + (z_2 + z1)$

2.3 Polar Representation

In this section, the polar form of a complex number has been discussed.

If $z = x + iy$

then we can write $z = \gamma \cos\theta + i\gamma \sin\theta$

where $\gamma = \sqrt{x^2 + y^2}$, $\cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$, $\sin\theta = \dfrac{y}{\sqrt{x^2+y^2}}$

$$|z| = \gamma = \sqrt{x^2 + y^2}$$

The argument of a complex number written as 'arg.z' is defined to be θ and its Principal value is that value of θ which lies between $-\pi$ and π.

2.4 The powers and roots of a complex number have been dealt with in this section. The Laplace-de-Moivre formula has been proved and with the help of the formula, the method for finding the nth root of a complex number has been evolved.

The content prescribed in the syllabus has been dealt with thoroughly in this chapter. However, following points should be kept in mind in order to avoid confusions and wrong notions:

(i) The number a+ib is not the sum of a and ib. This is a way of denoting a complex number.

(ii) In the number a+ib, only b is called its imaginary part and not ib.

(iii) The standard definition of a complex number should be a+ib where $a, b \in R$, $i = \sqrt{-1}$.

3. Learning Outcomes:

The followings are the essential learning outcomes for all:-

(i) The students should acquire the concept of a complex number and the relationship between the set of reals and the set of complex numbers. Complex numbers are an extension of real numbers.
They should be able to define and determine the real and imaginary parts of a complex number.

(ii) The students should be able to represent a complex number on Argand diagram and as an ordered pair.

(iii) The students should be able to perform the operations of addtion and multiplication on the set of complex numbers.

(iv) The students should learn the properties of the operations of addition and multiplication on complex numbers.

(v) The conjugate, modulus and argument of a complex number and the triangle inequality should be clear to the students.

They should learn certain rules connected with the moduli and conjugates.

(vi) The students should develop the ability to represent a complex number in the polar form (r,θ).

(vii) De-Moivre's formula should be clear to the students and they should be able to prove it.

(viii) The students should be able to extract nth roots of a complex number.

(ix) The students should be able to solve the questions on complex numbers as given in the text book.

It is also expected that higher ability group of the students develop a better understanding and insight into the complex numbers. They should be able to solve the problems which require more understanding and insight such as:-

(i) proving the uniqueness of identity for an operation, uniqueness of the inverse of an element.

(ii) finding the 4th, 5th roots of a complex number.

(iii) proving the formula for $\sin(\theta_1+\theta_2)$, $\cos(\theta_1+\theta_2)$, $\cos 3\theta$, $\sin 4\theta$ etc.

(iv) factorizing expressions like $x^2 + a^2$ etc.

4. Teaching Strategies:

It is needless to say that the students should be properly motivated for learning a certain concept and a good teacher will always try to motivate his students in order to infuse in them the readiness to learn. There may be numerous ways to do so and it depend largely on the skill and resourcefulness of the teacher in that how he does it. A good teacher should keep in mind that a person becomes motivated to do something if:-

(i) he sees a need for it

(ii) he gains an understanding of it.

As far as complex numbers are concerned, the need for these may be shown as follows by asking the students to solve the given equations:

$$2x - 6 = 0 \qquad (i)$$
$$2x + 6 = 0 \qquad (ii)$$
$$3x + 5 = 0 \qquad (iii)$$
$$x^2 - 9 = 0 \qquad (iv)$$
$$4x^2 - 25 = 0 \qquad (v)$$
$$x^2 - 3 = 0 \qquad (vi)$$
$$9x^2 + 1 = 0 \qquad (vii)$$
$$x^2 + 4 = 0. \qquad (viii)$$
$$25x^2 + 1 = 0 \qquad (ix)$$

If the replacement sets are taken to be N,Q,I and R respectively, the students will be able to see that

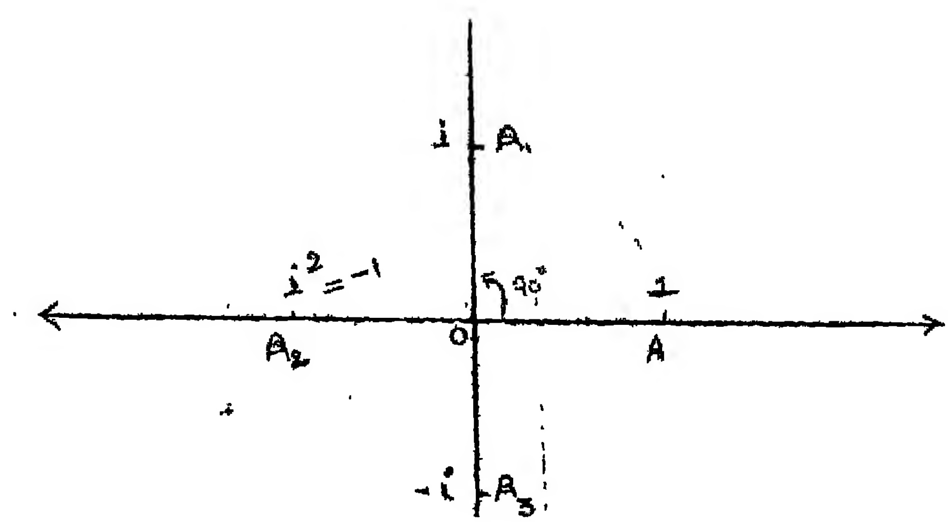
(a) when the replacement set is N, equation (i) has the solution and other equations can not be solved (only one solution x = 3 of (iv) belongs to N).

(b) when the solution set is I, equations (i),(ii) and (iv) are solvable.

(c) when the solution set is R, equations (i) to (vi) are solvable. Equations (vii)-(ix) cannot be solved in the field of real numbers because there are no real numbers whose squares are negative. The need for finding the square roots of negative numbers lead to the discovery of the complex numbers. The historical background given in the 'Introduction' may be used to motivate the students.

The students may be motivated by making them realize the importance of complex numbers by letting them know the part played by these numbers in the field of technology, engineering and probability.

The teacher may explain to the students the meaning of 'i' as an operator.



The operation 'i' consists of simply in turning OA through 90° in the anti-clockwise direction. Thus the operation 'i' sends A to A_1. This is clear that 'i^2' i.e. repeating the operation twice, will send A to A_2. i^3 will send it to A_3 and i^4 will bring it back to A again. Since $\overline{OA_2} = -\overline{OA}$, obviously $i^2 = -1$.

The students can also be motivated to find the series representing $\sin\theta$ and $\cos\theta$ by using the following facts:-

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots.$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

and $$a + ib = x + iy \Rightarrow a = x,\ b = y.$$

The sense of achievement works as a great force for motivation. It will be better if the teacher assigns some easy exercises from some of the additional exercises given at the end of this chapter, which are related to the concepts just taught. For example, when the addition on complex numbers has been defined, some problems on addition may be given to the students to solve.

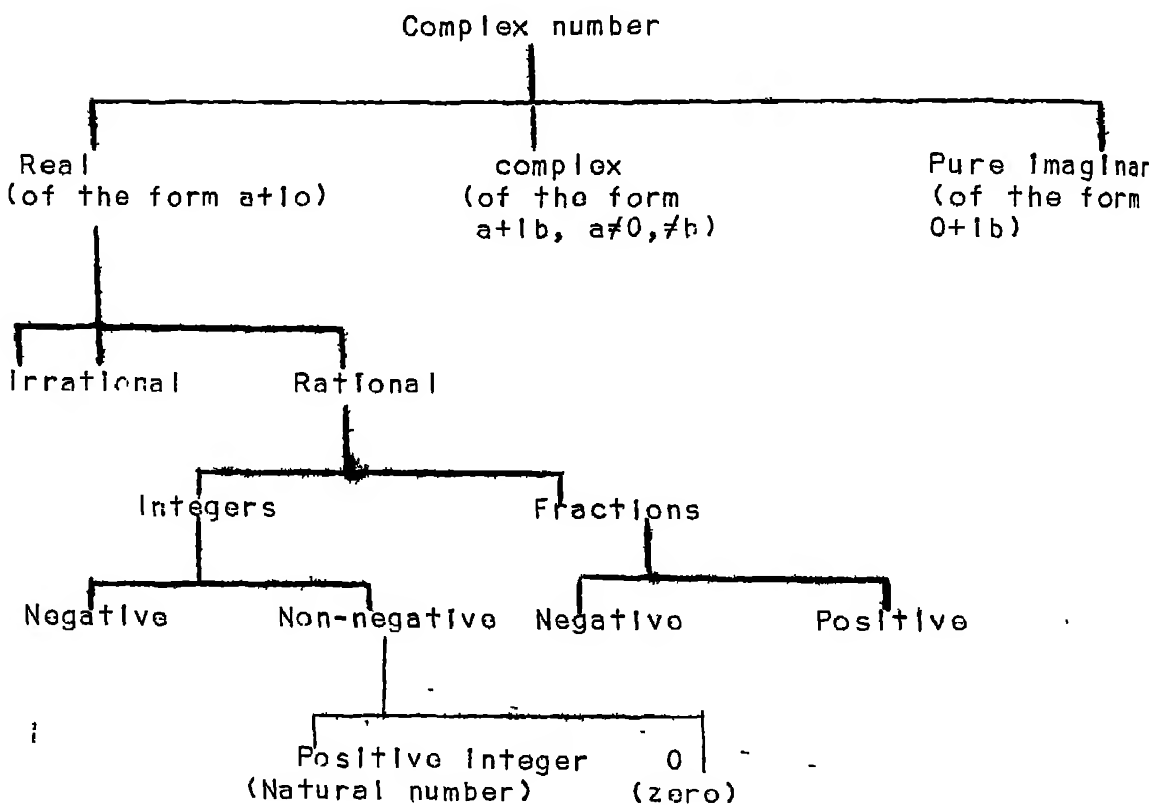
<u>Misconceptions and Common Errors:</u>

The following misconceptions are likely to take place. Care should be taken to remove them.

(i) a+ib is the sum of a real number a and an imaginary number ib. This concept is wrong because addition of real and imaginary numbers is not defined. Actually a+ib is one of the ways to represent a complex number whose real part is a and imaginary part is b.

(ii) In the number x+iy, sometimes it is mistaken that iy is the imaginary part. The imaginary part of this complex number is only 'Y' not iy.

(iii) Sometimes it is misconceived that real numbers are not complex numbers. It should be made clear that real numbers are those complex numbers whose imaginary parts are zero. Of course, real numbers cannot be imaginary numbers. The complex number system may be illustrated as follows:-



(iv) It is very likely for the students to think that the complex numbers are ordered. Actually, there is no order relation in the set of complex numbers. This precisely means that given two complex numbers z_1 and z_2, we cannot say which one is greater or less.

(v) The bijection between C and R×R should be explained explicitly as

(i) to every member of R×R, there corresponds a unique point in the xy-plane.

(ii) to every complex number, there corresponds a unique member of R×R. For example

$$3 + 2i \rightarrow (3,2)$$

(vi) The students generally commit mistake in using formulae (2.1) to (2.5) and (2.15). Sufficient practice of these should be done.

(vii) The rule $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ is true only when a and b be both are not negative real numbers. If both a and b are negative and this rule is applied, we get the incorrect result. The students are susceptible to this mistake. For example, to find the value of $\sqrt{-16} \times \sqrt{-9}$, we have

(i) $\sqrt{-16} \times \sqrt{-9} = 4i \times 3i = 12i^2 = -12$

(ii) $\sqrt{-16} \times \sqrt{-9} = \sqrt{144} = 12$

Here (i) is correct because

$$\sqrt{-a^2} = |a|i$$

(c) Additional Exercises:

I. Simplify

(a) $(3 + 4i) + (2 + 3i)$

(b) $(2 + 3i) - (\sqrt{2} + 5i) + i\sqrt{5}$

(c) $(2 + 5i) - (4 - i)$

(d) $(1 - 3i).(2 + 5i)$

(e) $(4 + 5i)(2 - 3i)$

(f) $(5 + 2i)^3$

(g) $(3 - 2i)\{(2-4i) + (5+2i)\}$

II. Solve for x and y:

(a) $x + iy = 3-4i$

(b) $x + iy - (3+2i) = 5-6i$

(c) $3x + 2y + 1 + (x+2y)i = 0$

(d) $x + 2yi = i + (-2 + 5i)$

III. Write in the standard form $a + ib$

(a) $\frac{3+4i}{2i}$ (b) $\frac{1-i}{1+i}$

(c) $\frac{1+i}{1-i}$ (d) $\frac{2+3i}{2-3i}$

(e) $\frac{3-5i}{4+3i}$

IV. Solve each of the following for z and $\bar{z}$:-

(a) $z\bar{z} + 2(z-\bar{z}) = 25-12i$

(b) $z\bar{z} - 3(z-\bar{z}) = 4+6i$

V. Solve for x and y:

(a) $(1,1).(x,y) = (1,3)$

(b) $(2,5).(x,y) = (1,12)$

(c) $(4,0).(x,y) = (-11,-13)$

VI. Prove each of the following where $z,w \in C$:

(a) $|wz| = |w||z|$

(b) $|z| = \sqrt{z\bar{z}}$

(c) $|w|+|z| \geq |w+z|$

(d) $|1+z| \leq 1 + |z|$.

VII For each of the followings, find (i) $z_1 z_2$ and (ii) $\frac{z_1}{z_2}$:

(a) $z_1 = 2-2i$, $z_2 = -1+i$

(b) $z_1 = 3i, \quad z_2 = 2-5i$

(c) $z_1 = 2(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}), \quad z_2 = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$

VIII. Write the multiplicative inverses of the followings:

(a) $2+i$ (b) $3-4i$ (c) $x+iy$

IX. Express the followings in the form $a+ib$:

(a) $3(\cos 30^\circ + i \sin 30^\circ)$

(b) $4(\cos 45^\circ + i \sin 45^\circ)$

(c) $2(\cos 120^\circ + i \sin 120^\circ)$

(d) $\frac{1}{2}(\cos 60^\circ + i \sin 60^\circ)$

(e) $5(\cos 0^\circ + i \sin 0^\circ)$

X. Use De-Moivre's Theorem to compute the followings:

(a) $(1+i)^7$ (b) $(-1-i)^5$ (c) $(\sqrt{3}-i)^8$

(d) $(-i)^{12}$ (e) $(-\frac{1}{2} + \frac{\sqrt{3}}{2} i)^6$

(f) $(-1+i)^{10}$ (g) $(\cos 36^\circ + i \sin 36^\circ)^5$

(h) $[2(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12})]^6$

(i) $[3(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2})]^4$

XI. Show that $(2+\sqrt{-121})^{1/3} + (2-\sqrt{-121})^{1/3}$ is a real number.

XII. Write the conjugate of $3(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})^3$.

XIII. Write the product $\sqrt{-a^2} \cdot \sqrt{-b^2}$.

<u>Solutions/Hints For The Difficult Problems:</u>

<u>Exercise 2.1</u>. Q.3: The answer given in the text book is $\frac{1+i}{1-i}$. This may be put in the standard form by making the denominator real, i.e.

$$\frac{1+i}{1-i} = \frac{(1+i)^2}{1^2 - i^2}$$

$$= \frac{1+2i + i^2}{1+1}$$

$$= \frac{2i}{2}$$

$$= i$$

Miscellaneous Exercise:

Q.1 Let $z = x+iy$; then $|z| = \sqrt{x^2+y^2}$.

Now if $z = 0$, then $x = 0$, $y = 0$ so that $|z| = 0$. Conversely, assume that $|z| = 0$; then $\sqrt{x^2+y^2} = 0 \Rightarrow x^2+y^2 = 0 \Rightarrow x = 0, y$

$\Rightarrow z = 0 + i0 = 0$.

Q.4 Refer to the figure on page 28 in the text book. In the Figure, O represents the complex number z_1-z_2. Therefore $OQ = |z_1-z_2|$. Since OQP_1P_2 is a parallelogram, $OQ = P_1P_2$. Therefore, $P_1P_2 = |z_1-z_2|$.

Q.8 $|z+2-i| = 4$

$\therefore \quad |z-(i-2)| = 4$

Therefore the distance between the points representing z and $(i-2)$ is 4 (refer to Q.4). Also $i-2$ is a fixed point. Hence z lies on a circle of radius 4 with centre at $i-2$.

Q.9 Let $\sqrt{-15-8i} = x+iy$

$\therefore \quad (x+iy)^2 = -15-8i$

or $\quad x^2-y^2 + 2ixy = -15-8i$.

Equating real and imaginary parts, we have

$$x^2 - y^2 = -15 \qquad (1)$$

$$2xy = -8 \qquad (2)$$

Now $$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$$

$$= 225 + 64$$

$$= 289$$

$$x^2 + y^2 = 17 \qquad (3)$$

from (I) and (III),

$$x^2 = 1 \quad \text{or} \quad x = \pm 1$$
$$y^2 = 16 \quad \text{or} \quad y = \pm 4$$

According to (II), product xy is negative; therefore either

$$x = 1, \quad y = -4$$

or, $$x = -1, \quad y = 4$$

Therefore the required square root is

$$-1 + 4i \quad \text{or} \quad 1 - 4i$$

Thus $$\sqrt{-15 - 8i} = \pm 1 \mp 4i.$$

Q.10 Let A, B and P represent the complex numbers 3, -3 and z respectively on the Argand diagram.

Given that

$$\left| \frac{z-3}{z+3} \right| = 2$$

i.e. $$\frac{|z - 3|}{|z-(-3)|} = 2$$

Hence $$\frac{PA}{PB} = \frac{2}{1}$$

This is the locus of a point P(z) which moves in such a manner that the ratio of its distance from A(3,0) to its distance from B(-3,0) is 2.

We shall now show that this locus is a circle. Taking $z = x+iy$ we have

$$\left| \frac{(x-3) + iy}{(x+3) + iy} \right| = 2$$

or $$\frac{\sqrt{(x-3)^2 + y^2}}{\sqrt{(x+3)^2 + y^2}} = 2$$

or $(x-3)^2 + y^2 = 4(x+3)^2 + 4y^2$

or $x^2-6x+9+y^2 = 4x^2 + 24x + 36 + 4y^2$

or $3x^2+30x + 3y^2 + 27 = 0$

or $x^2 + 10x + y^2 + 9 = 0$

or $x^2 + 10x + 25 + y^2+9 - 25 = 0$

or $(x+5)^2 + y^2 = 16$

This is the equation of a circle with centre at $(-5,0)$ and radius 4.

Note: This problem should be solved by the teacher after finishing Chapter 11.

Teaching Aids

Films on complex numbers of 15/20 minutes' duration are available in the market. These can be used as audiovisual aids before introducing the formal lessons on complex numbers. Some charts may be prepared to indicate properties of mathematical operations on complex numbers.

(g) Projects and Investigatory Problems:

Students may be encouraged to take the projects such as finding pairs of complex numbers so that their sum is equal to

their product or finding the pairs of points on the unit circle such that distance between them is a rational number.

(k) Some Challenging Problems

(a) Use De Moivre's Theorem to find the trigonometric identities for $\sin 3\theta$, $\cos 3\theta$, $\sin 4\theta$, $\cos 4\theta$.

(b) Prove that the points with coordinates $(\cos 60^\circ + i \sin 60^\circ)^n$ for $n = 1,2,3,4,5,6$ are the vertices of a regular hexagon inscribed in an unit circle.

(c) Find the fifth root of unity.

(d) Show that multiplicative inverse of a nth roots of 1 is also a nth root of 1.

Chapter Test

(I) Oral Questions

(i) What are the moduli of $2-i$, $3+4i$, $p+qi$?

(ii) What are the conjugates of $7+4i$, $-3i-2$?

(iii) If $x+iy = 7-4i$, what are the values of x and y?

(iv) If

a) $\overline{(z_1+z_2)} = 2+3i$; what is $\bar{z}_1 - \bar{z}_2 = $?

b) $\overline{\left(\frac{z_1}{z_2}\right)} = -5+i$, what is $\frac{\bar{z}_1}{\bar{z}_2}$?

(v) What are the 'Arg' of $\sqrt{3}+i$, $1+i$?

(vi) $(\cos 30^\circ + i \sin 30^\circ)^2$ is equal to what?

(vii) $e^{i3\theta}$ is equal to what?

(II) Written Test

Time: 1 hour
M.M.: 25

3. Change $2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ in the standard form.

4. What is the modulus of the complex number, $(-2\sqrt{6} + 2\sqrt{2}i)$?

5. Use De Moivre's Theorem to compute $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^6$

6. Find the square root of

7. Find x and y such that

$$(2x-3y-6) + (x-4y)i = 0$$

8. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

prove that

$$z_1 z_2 = r_1 r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}.$$

Show it geometrically.

9. Express $\sqrt{3}-i$ in polar form.

10. Find all the sixth roots of unity.

...

ANSWER

Additional Exercise (Answers)

1.(a) $6+7i$

(b) $(2-\sqrt{2}) - (2-\sqrt{5})i$

(c) $-2 + 6i$

(d) $17-i$

(e) $23 - 2i$

(f) $65 + 142i$

(g) $17 - 6i$

2. Value of x and y

(a) $\frac{x}{3}$ $\frac{y}{-4}$

(b) 8 -4

(c) $-\frac{1}{2}$ $+\frac{1}{4}$

(d) -2 , 3

3. Standard form a+ib

(a) $2 + \frac{3}{2}i$

(b) $0 - i$

(c) $0 + i$

(d) $-\frac{5}{13} + \frac{12}{13}i$

(e) $-\frac{3}{25} - \frac{29}{25}i$

4.(a) $z = 4 - 3i$

$\bar{z} = 4 + 3i$

(b) $z = \sqrt{3}-i$

$\bar{z} = \sqrt{3}+i$

7. $z_1 z_2$ z_1/z_2

(a) $4i$ 2

(b) $15+6i$ $-\frac{5}{3}+\frac{2}{3}i$

8. Multiplicative Inverse

(a) $\left(\frac{2}{5}, -\frac{1}{5}\right)$

(b) $\left(\frac{3}{25}, \frac{4}{25}\right)$

(c) $\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$

9. Form of a+ib

(a) $\frac{3\sqrt{3}}{2}+\frac{\sqrt{3}}{2}i$

(b) $(2\sqrt{2})+i(2\sqrt{2})$

(c) $-1+i(\sqrt{3})$

(d) $\frac{1}{4}+\frac{\sqrt{3}}{4}i$

(e) $5+0i$

10. Using De Moiver's Theorem and computing value

(a) $2^{7/2}[\cos 7(2r\pi+\frac{\pi}{4})+i\sin(2r\pi+\frac{\pi}{4})]$, $r=0,1,2,\ldots,6$

(b) $-2^{5/2}[\cos 5(2r\pi+\frac{\pi}{4})+i\sin 5(2r\pi+\frac{\pi}{4})]$, $r=0,1,2,\ldots,4$

(c) $2^{8}[\cos 8(2r\pi+\frac{\pi}{6})+i\sin 8(2r\pi+\frac{\pi}{6})]$, $r=0,1,2,\ldots,7$

(d) $[\cos 12(2r\pi+\frac{\pi}{2})-i\sin 12(2r\pi+\frac{\pi}{2})]$, $r=0,1,\ldots,11$

(e) $[\cos 6(2r\pi+\frac{2\pi}{3})+i\sin 6(2r\pi+\frac{\pi}{3})]$, $r=0,1,\ldots,5$

(f) $[\cos 10(2r\pi+\pi)+i\sin 10(2r\pi+\frac{\pi}{2})]$, $r=0,1,2,\ldots,9$

(g) $[\cos 5(2r\pi+\frac{2\pi}{10})+i\sin 5(2r\pi+\frac{2\pi}{10})]$, $r=0,1,2,\ldots,4$

(h) $2^6[\cos 6(2r\pi + \frac{5\pi}{12}) + i \sin 6(2r\pi + \frac{5\pi}{12})]$, $r = 1,2,\ldots,5$

(i) $3^4[\cos 4(2r\pi + \frac{3\pi}{2}) + i \sin 4(2r\pi + \frac{3\pi}{2})]$, $r = 1,2,3,.$

12. $3(\cos \frac{3\pi}{6} - i \sin \frac{3\pi}{6})$

13. $-ab$

Chapter Test (Answers)

(I) Oral Question

1. a) $\sqrt{5}$ (b) 5 (c) $\sqrt{p^2+q^2}$

2. $7-4i$, $-2+3i$

3. $x = 7$, $y = -4$

4. a) $2-3i$ (b) $-5+i$

5. (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$

6. $\frac{1+i\sqrt{3}}{2}$

7. $(\cos 3\theta + i \sin 3\theta)$

(II) Written Test

1. $\left(\frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}\right)$

2. $(49 - 10i)$

3. $(1 + \sqrt{3}\, i)$

4. $4\sqrt{2}$

5. $[\cos 6(2r\pi + \frac{2\pi}{3}) + i \sin 6(2r\pi + \frac{\pi}{3})]$, where $r = 0,1,2,\ldots,5$

7. $x = \frac{24}{5}$, $y = \frac{6}{5}$

9. $2(\cos \frac{\pi}{6} - i \sin \frac{\pi}{6})$

10. 1, $\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$, $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$, $\cos \pi + i \sin \pi$, $\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$, $\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}$.

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3. Modern Introductory Mathematics by Wendell G.Johnson)Mc Graw Hill Book Company and Franke N. Zaccaro) 1966.

4. Complex Numbers by Walter Lederman (Library of Mathematics published by Routledge and Keqanpaul Lt. 1967).

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CHAPTER 3

QUADRATIC EQUATIONS

1. INTRODUCTION

Students have come across quadratic polynomials in their earlier classes and are also familiar with, how to factorise a quadratic polynomial. An equation whose left hand side is a quadratic polynomial $ax^2 + bx + c = 0$ $(a \neq 0)$ and right hand side is zero is called a quadratic equation.

Many quadratic equations cannot be solved by factorisation method. Thus factorisation method is not a general method for finding solutions of quadratic equations. For example, $x^2 + x + 1 = 0$ cannot be solved by factorisation method. So the need for having a general method of solving a quadratic equation arises.

From ancient times, mathematicians have used various techniques to solve a quadratic equation in one unknown variable. The ancient Greeks used geometrical methods because of their logical difficulties with irrational and cumbersome nature of Greek numerals. The Hindus and Arabs used a method very similar to the 'completing the square method' as given in the text. The method was first discovered in 11th century by Indian mathematician 'SRIDHARACHARYA'. The discovery of complex numbers lead to the acceptance of complex roots for a quadratic equation. In this chapter, a general formula for finding the roots of a quadratic equation is arrived at and the relationships between the roots and the coefficients of the equation are discussed. Some application aspects of the quadratic equations are also dealt with.

The knowledge of quadratic equation and its solution is necessary and essential in many branches of mathematics like coordinate geometry, trigonometry, complex numbers etc.

2. CONTENT ANALYSIS

In this section, the number of each subsection is in accordance with the text-book.

2.1 Solution of quadratic equation

(1) The general form of a quadratic equation is

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

and the solutions or the roots of the equation are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(2) It is important that $a \neq 0$, for otherwise the polynomial $ax^2+bx+c = 0$ does not remain a quadratic one.

(3) The expression $\Delta \equiv b^2-4ac$ is called the discriminant for the equation $ax^2 + bx + c = 0$ (§3.1). It is called so, because considering the nature of this expression, the nature of the roots are discriminated. Similarly, if α and β are roots of $ax^2 + bx + c = 0$ $(a \neq 0)$, then

(i) α and β are real and distinct if $\Delta > 0$

(ii) $\alpha = \beta$, if $\Delta = 0$

(iii) α and β are complex numbers (imaginary) if $\Delta < 0$.

2.2 Symmetric Functions of Roots

If α and β are roots of $ax^2 + bx + c = 0$, $\alpha + \beta$ and $\alpha\beta$ are called elementary symmetric functions, because all symmetric functions of α and β are expressible as functions of these elementary symmetric functions. Examples of symmetric functions as functions of elementary symmetric functions are given in the text-book. $\alpha+\beta$ and $\alpha\beta$ are symmetric in the sense that they remain unaltered by interchanging α and β. Expressions for $\alpha + \beta$ and $\alpha\beta$ in terms of coefficients of $ax^2 + bx + c = 0$ are also given in the textbook.

2.3 Graph of a Quadratic Polynomial

The graph of the quadratic polynomial $y = ax^2 + bx + c$, $(a \neq 0)$ is a parabola. The roots of the polynomial are given by the x-coordinates of the points where the parabola intersects the x-axis.

2.4 Problems

Quadratic equations have important uses. Sometime, we can easily solve a biquadratic equation by reducing it to a quadratic equation. Quadratic equations are often used in solving word problems from everyday life and other branches of science.

3. LEARNING OUTCOMES

After studying this chapter, the students are expected to -

(1) differentiate between a quadratic polyhomial and a quadratic equation;

(2) adopt a correct method for solving a quadratic equation;

(3) find solution of a quadratic equation;

(4) find relationship between the roots and the coefficients of the given equation;

(5) draw the graphs of the quadratic polynomials;

(6) solve different types of problems leading to quadratic equations.

4. TEACHING STRATEGIES

In this section the number of each subsection is in accordance with the text book.

3.4 Problems

To make the teaching of quadratic equations more interesting, it is better to start the teaching with a word problem, e.g. the following problem is given in the text (Ex. 3.9).

"A two-digit number is four times the sum and three times the product of its digits. Find the number".

Here the teacher should emphasise that once the two digits are fixed the required number is known. So the two digits are denoted by two symbols x and y which have to be evaluated so that the conditions of the problem are satisfied. In this connection, it will be worth while to point out, what is the role of an unknown variable in an equation and the distinction between an equation and an identity.

In the above word problem, the required equation has to be got from the students by proper questioning. In this particular case, the teacher may show that the problems leads to the quadratic equation $6x^2-12x = 0$, where x is the number in the tens place in the digit. This equation can be solved by factorising the polynomial $6x^2-12x$. But there are quadratic equations which cannot be solved by factorisation method e.g. $2x^2 + 14x + 5 = 0$.

3.1 Solution of Quadratic Equations

To solve a general quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$, a formula for the roots has to be given by the method of 'completing the square' as given in the text.

The teacher may solve a numerical problem by using completing the square method'. Then he may ask the students to solve the general equation $ax^2 + bx + c = 0$ by using the same method so that the students will arrive at the formula for the roots. At this stage, students may be advised to solve numerical problems for practice.

3.4 Problems

After sufficient practice in solving numerical exercises, students may be encouraged to solve word problems. In dealing with word problems, the teacher may see that the students develop the skill of translating the verbal language into the appropriate mathematical symbols.

3.3 Graph of a Quadratic Polynomial

The use of quadratic equation in mechanics can be stressed through the fact that the path of a projectile is a parabola whose equation may be written in the form $y = f(x)$, where $f(x)$ is a quadratic polynomial in x.

Special Cases

Sometimes while solving an equation with radical symbol, we square both the sides of the equation to get rid of radicals and reduce the equation into the form $ax^2 + bx + c = 0$ $(a \neq 0)$. This process may bring in some solutions of the given equation which, in fact, do not satisfy the solution. These are extraneous solutions

This is because, conventionally by $\sqrt{a}$, we mean $+\sqrt{a}$, but while squaring $\sqrt{a}$, we implicitly admit both the values of $\sqrt{a}$, i.e $+\sqrt{a}$ and $-\sqrt{a}$. In this way we bring in some extraneous solutions The following examples will clarify the situation:

<u>Example 1:</u> Solve: $x = 2+ \sqrt{2x -1}$ (1)

$$\text{i.e., } x-2 = \sqrt{2x-1}$$

$$\text{i.e., } x^2-4x + 4 =2x-1$$

$$\text{i.e., } (x-5)(x-1) = 0$$

$$\text{i.e., } x = 5 \text{ or } 1$$

But $x =1$ does not satisfy the equation (1).

Hence, $x = 5$ is the only root of the equation. In this case, $x = 1$ is an extraneous root.

<u>Example 2:</u> Solve: $x = 2 - \sqrt{2x-5}$

$$\text{Then } x-2 = -\sqrt{2x-5}$$

$$\text{i.e., } x^2-4x+4 = 2x-5$$

$$\text{i.e., } x^2-6x +9 = 0$$

$$\text{i.e., } (x-3)^2 = 0$$

$$\text{i.e., } x = 3$$

But $x = 3$ does not satisfy the equation and so, it is not a solution of the equation.

<u>Hints and Solutions to Difficult Problems in Exercises 3.1 (Chapter 3)</u>

3. $\sqrt{x} = x-2$

$$\text{i.e., } x = x^2 - 4x+4$$

i.e., $x^2 - 5x + 4 = 0$

i.e., $(x-4)(x-1) = 0$

i.e., $x = 4$ or 1

But $x = 1$ does not satisfy the equation. So $x = 4$ is the only solution.

05. Let α, β be the roots of the given equation.

Then
$$\left.\begin{aligned} \alpha+\beta &= -(2a+1) \\ \alpha\beta &= a^2 + 2 \end{aligned}\right\} \qquad (1)$$

Let $\beta = 2\alpha$, then (1) becomes

$$3\alpha = -(2a+1)$$

$$2\alpha^2 = a^2 + 2$$

leading to $2\left(\frac{2a+1}{3}\right)^2 = a^2 + 2$

i.e., $(a-4)^2 = 0$

which gives $a = 4$.

06. Let the length of the cloth be x meters.

Then from the given data

$$(x+4)\left(\frac{35}{x} - 1\right) = 35$$

which gives $x = -14$ and $x = 10$.

Since, length cannot be negative, hence take $x = 10$

$\therefore$ length of the cloth = 10 m.

Q7. Let x = number of students in the group. Then the share of each student = p/x, where p = the price of the tape-recorder in rupees.

From the given data

$$p/x + 1 = p/x-2 \qquad (1)$$

i.e., $\frac{p}{x-2} - \frac{p}{x} = 1$

On solving we get $p = \frac{x^2 - 2x}{2}$ (2)

Now, the price is in between Rs. 170 and Rs. 195.

$\therefore$ we have

$$170 \leq p \leq 195$$

i.e., $170 \leq \frac{x^2 - 2x}{2} \leq 195$

i.e., $340 \leq x^2 - 2x \leq 390$

i.e., $341 \leq x^2 - 2x + 1 \leq 391$

i.e., $341 \leq (x-1)^2 \leq 391$

This $\Rightarrow 324 < 341 \leq (x-1)^2 \leq 391 \leq 400$

$\Rightarrow 324 < (x-1)^2 < 400$

$\Rightarrow 18^2 < (x-1)^2 < 20^2$

$\Rightarrow 18 < x-1 < 20$

$\Rightarrow 19 < x < 21$

$\Rightarrow x = 20$

$\therefore$ x = 19 is the only positive integer satisfying this inequality.

Substituting $x = 20$, in (2) we have

$$p = 180$$

So, the price of the tape-recorder is Rs. 180.

$$\frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} = \frac{a}{x}$$

By componendo and dividendo, we get

$$\frac{\sqrt{a+x}}{\sqrt{a-x}} = \frac{a+x}{a-x}$$

Squaring both sides and simplifying we get

$$x[2a^2 - a^2 - x^2] = 0$$

i.e., $x = 0$ and $x = \pm a$

Let x be the percentage of increase in output at one year and 'a' be the output before 2 years. Then from the data of the problem

$$a + \frac{ax}{100} + (a + \frac{ax}{100}) \frac{x}{100} = 2a$$

i.e, $1 + \frac{x}{100} + (1 + \frac{x}{100}) \frac{x}{100} = 2$

Put $y = \frac{x}{100}$, then we have

$$1+y + (1+y)y = 2$$

i.e., $y^2 + 2y - 1 = 0$

i.e., $y = (-1 \pm \sqrt{2})$.

Here $y = -1 -\sqrt{2}$ is not admissible, as y has to be greater than 0.

So $y = -1 + \sqrt{2}$

i.e. $x = 100(\sqrt{2} - 1)\%$

012 Let the number of points marked on the plane be n. Then the number of the line segments $= \frac{n(n-1)}{2}$

then $\frac{n(n-1)}{2} = 10$

i.e., $n^2 - n - 20 = 0$

i.e., $n = -4, 5$

But n being a positive integer, so $n \neq -4$. Hence, $n = 5$. Thus 5 points are marked on the plane.

Note: As the number of line segments $= {}^nC_2$ the problem should be solved after Chapter 5.

Let α, β be the roots of the equation

then $\left.\begin{aligned} \alpha+\beta &= -p \\ \alpha\beta &= 45 \end{aligned}\right\}$ (1)

Squared difference of roots $= (\alpha-\beta)^2 = 144$. (1) gives

$$144 = (\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$$

$$= (-p)^2 - 4.45$$

$$= p^2 - 180$$

giving $p = \pm 18$

Now, we have two sets

$$\begin{aligned} \alpha+\beta &= 18 \\ \alpha\beta &= 45 \end{aligned} \quad \text{and} \quad \begin{aligned} \alpha+\beta &= -18 \\ \alpha\beta &= 45 \end{aligned}$$

giving $\alpha+\beta = 18$ and $\alpha+\beta = -18$

$\alpha-\beta = 12$ $\quad$ $\alpha-\beta = -12$

leading to

$\alpha = 15$ and $\alpha = -15$

$\beta = 3$ $\quad$ $\beta = -3$

Q14 We have $\alpha+\beta = 6$ (1)

$\alpha\beta = a$ (2)

$3\alpha + 2\beta = 20$ (3)

From (1) and (3) $\alpha = 8$ and $\beta = -2$

From (2) $a = -16$

Misconceptions/Common errors

Some of the common errors committed by the students are:

(1) In writing down the values of the coefficients a, b and c from the given equation $ax^2 + bx + c = 0$, especially with respect to negative sign.

(2) In translating the word problems to equational form i.e. reducing statements to equations.

(3) In writing a two digit number whose digits in tens and units places are x and y respectively, students often write the number as xy or x+y in place of 10x+y.

(4) In writing the formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

as $x = \frac{-b \pm \sqrt{b^2-4ac}}{2}$

or as $x = \frac{-b \pm \sqrt{b^2-4ac}}{2}$

Proofs of Important Theorems Left Out in the Text Book

I. "In a polynomial, equation with real coefficients, complex roots occur in conjugate pairs".

Proof: Let $f(x) = 0$ be the given polynomial equation of degree n and $\alpha + i\beta$ be a complex root of this equation. We will prove that $\alpha - i\beta$ is another root. We have

$$(x-\alpha-i\beta)(x-\alpha+i\beta) = (x-\alpha)^2 + \beta^2 .$$

Let $f(x)$ be divided by $(x-\alpha)^2 + \beta^2$, where deg. $f(x) = n$.

Then $f(x) = q(x)\{(x-\alpha)^2 + \beta^2\} + cx+d$, where deg. $q(x) = n-2$.

We have

$$f(\alpha+i\beta) = q(\alpha+i\beta)\{(\alpha+i\beta-\alpha)^2 + \beta^2\} + c(\alpha+i\beta)+d$$

$$= c(\alpha+i\beta) + d.$$

But $f(\alpha+i\beta) = 0$, since $\alpha+i\beta$ is a root of $f(x) = 0$.

So $c(\alpha+i\beta) + d = 0$.

Now, equating real and imaginary parts of this equality.

We get $c\alpha + d = 0$ and $c\beta = 0$ giving $c = d = 0$ (Because $\alpha \neq 0$ and $\beta \neq$

So $$f(x) = q(x)\{(x-\alpha)^2 + \beta^2\}$$

$$= q(x)(x-\alpha-i\beta)(x-\alpha+i\beta),$$

i.e. $\alpha - i\beta$ is a root of $f(x) = 0$.

II. If $ax^2 + bx + c = 0$ has three distinct root, then $a = b = c =$

Proof: Let α, β, γ be three distinct roots of $ax^2 + bx + c = 0$. Then

$$a\alpha^2 + b\alpha + c = 0$$

$$a\beta^2 + b\beta + c = 0$$

and $$a\gamma^2 + b\gamma + c = 0$$

From the set (1), we have,

$$a(\alpha^2-\beta^2) + b(\alpha-\beta) = 0 \text{ etc.}$$

i.e. $(\alpha-\beta)\{a(\alpha+\beta)+b\} = 0$, etc. (2)

Thus we have

$$a(\alpha+\beta) + b = 0, \; a(\beta+\gamma) + b = 0, \; a+(\gamma+\alpha) + b = 0 \qquad (3)$$

Since $\alpha-\beta \neq 0$ etc.,

By subtraction in set (3), we have

$$a(\alpha-\beta) = 0 \text{ etc.} \qquad (4)$$

Since α, β, γ are distinct, therefore from (4) we have

$$a = b = c = 0$$

Note: As mentioned in the text book, $a = b = c = 0$ is a trivial case for which any $x \in R$ is a solution of $ax^2+ bx+c = 0$. Thus a quadratic equation cannot have more than two distinct roots.

5. CHAPTER TESTS

(a) ORAL TEST

(1) In the equations $3x^2+4x + 7 = 0$ and $x^2-x+1 = 0$

i) What is the sum of the roots ?

ii) What is the product of the roots?

iii) Which of the two equations have reciprocal roots?

(2) In each of the equations

i) $x^2 + 5x + 3 = 0$

ii) $x^2 - 7x + 4 = 0$

iii) $x^2 - x - 1 = 0$

and iv) $x^2 + px - q = 0$

What is the value of the discriminant?

(3) In the quadratic equation

$$ax^2 + bx + c = 0$$

i) When are the two roots equal?

ii) When are the roots real?

iii) When are the roots complex?

iv) When are the roots reciprocal?

(4) Let S and P denote respectively the sum and product of two roots of a quadratic equation. Give your comments about the nature of the roots in each of the following cases:

i) $S > 0$ and $P > 0$;

ii) $S > 0$ and $P < 0$;

iii) $S < 0$ and $P > 0$;

iv) $S < 0$ and $P < 0$.

(5) If $a = b = c$ in the quadratic equation $ax^2 + bx + c = 0$, what is the nature of its roots?

(6) What type of roots does the equation $ax^2 + bx + c = 0$ have when $b = 0$?

(7) To which type of equation does $ax^2 + bx + c = 0$ reduce when $a =$

(b) <u>WRITTEN TEST</u>

(1) Solve $3x^2 + 5x - 7 = 0$

(2) Solve $\frac{x+2}{x+3} - \frac{x^2}{x^2-9} = 1 - \frac{x-1}{3-x}$. What restriction has to be put on the values of x so that the equation makes sense?

3) Solve $\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = \frac{13}{6}$

4) Solve $|x^2+z| + |x| = 1$

5) Find the values of $\alpha^3 + \beta^3$ if α and β are the roots of $5x^2 + 6x + 1 = 0$.

(6) A tank can be filled in 4 hours by two pipes when both are used. How many hours are required for each pipe to fill the tank alone if the smaller pipe requires 3 hours more than the larger one. Compute answer to two decimal places.

(7) The sum of a number and its reciprocal is 10/3. Find the number.

(8) Find the base and height of a triangular field with an area of 2 square mts, if its base is 3 mt longer than its height.

(9) If P rupees are inverted at r% compounded annually, at the end of 2 years the amount will be $A = P(1+r)^2$. At what interest rate will Rs. 1000/- increase to Rs. 1440/- in 2 years?

(10) Form the equation with integral coefficients where roots are $\frac{5}{6}$ and $\frac{-4}{9}$.

Answers:

1. $\frac{-5 \pm\sqrt{109}}{6}$ 2. $\frac{-3\pm \sqrt{57}}{4}$, the restriction on x, $x \neq \pm 3$

3. $\frac{9}{13}$, $\frac{4}{14}$ 4. The solution set is ϕ.

5. -198 6. $\frac{5+ \sqrt{73}}{2}$ hrs & $\frac{5+ \sqrt{73}}{2}$ + 3 hours

i.e. 6.77 hrs and 9.77 hours

7. 3, $\frac{1}{3}$ 8. height = 1 mt, base = 4 mts

9. r = 1/5 10. $54x^2 - 21x-20 = 0$.

6. ADDITIONAL READING MATERIAL FOR ENRICHMENT: GRAPH OF A QUADRATIC EQUATION

Example 1: Construct the graph of the function

$$y = x^2 + 2x + 3$$

Sol.

$$y \equiv x^2 + 2x+3 = (x+1)^2 + 2.$$

So the vertex of the parabola is C(-1,2) and (0,3) is the point of intersection of the parabola and the y-axis. The branches of the parabola are directed upward as shown in the figure given below:

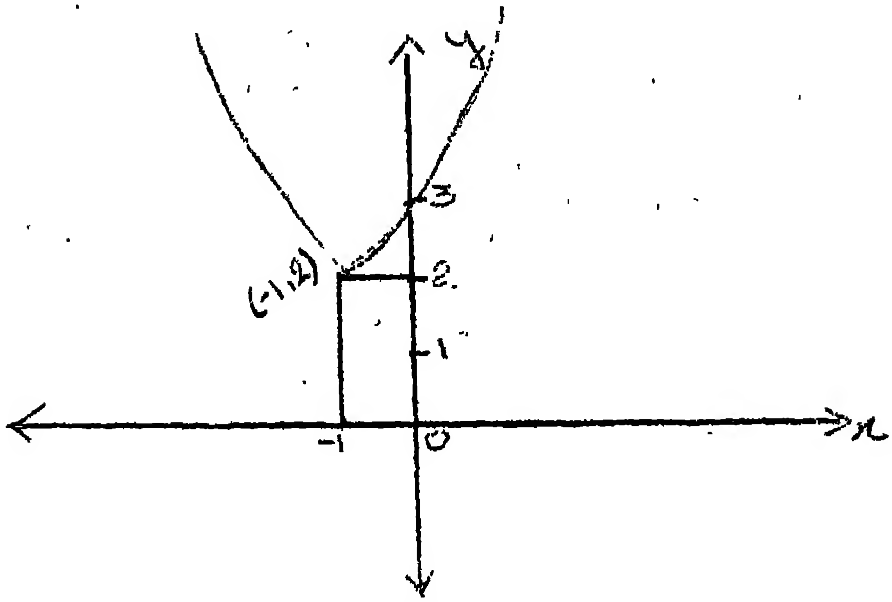

Fig. 3.1: Graph of $y = x^2+2x+3$

Exercises

Draw the graph of the following functions

1. $y = -2x^2 + 4x+1$
2. $y = -2(x-1)(x+3)$
3. $y = x^2 + x+1$
4. $y = 5x^2 + 3x+2$

Difference Between Inequalities and In equations

We note that order relation exists in the set of real numbers i.e. if a and b are two real numbers, only one of the following is tr

$$a < b$$

$$a = b$$

$$a > b$$

Note, the following relations:-

(i) $3 < 7$ (ii) $1 > 0$ (iii) $-4 < 1$ (iv) $x < 3$

(v) $4x > -6$ (vi) $2y < 6$

The, first three relations containing the known numbers are called inequalities while in the last three which contains unknown are called in equations.

Solution Sets of Quadratic Inequations (Interval method)

Here we will consider the solution set of inequations of the form $x^2 + bx + c > 0$ or $x^2+bx+c < 0$, where $x^2 + bx + c \equiv (x-\alpha)(x-\beta)$, α, β being real.

The following examples are self explanatory and will clarify the situation.

Ex. 1: Consider $y \equiv (x-2)(x-5) > 0$.

Here $y > 0$ only when $x-2$ and $x-5$ are both positive or both negative. This means that y will be positive only when $x < 2$ or $x > 5$ So the graph of the solution set of this inequations can be shown on the real line as follows.

Fig. 3.2

Here, the solution set is shown by thick line ——having breaks at $x = 2$ and $x = 5$.

The solution set is $\{x|x \in (-\infty,2) \cup (5,\infty)\}$

Ex.2. Consider $y \equiv (x-2)(x-5) < 0$.

Here $y < 0$ only when out of $(x-2)$ and $(x-5)$, one is positive and the other is negative. This can happen only when $x \in (2,5)$.

The solution set of this inequations on the real line is as shown below:

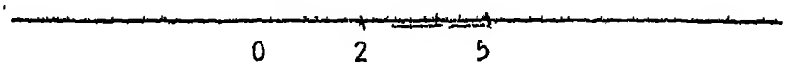

Fig. 3.3

Here the solution set is $\{x|x \in (2,5)\}$.

Exercises

Find the solution sets of following inequations and also represent the solution sets on the real line.

1. $(x+2)(x+5) < 0$
2. $(x+2)(x+5) > 0$
3. $(x-2)(x+5) > 0$
4. $(x-2)(x+5) < 0$
5. $(x+2)(x-5) > 0$
6. $(x+2)(x-5) < 0$

Additional Problems with Hints and Answers

Ex.1 Solve $|2x-3| = |x+7|$

The statement $|f(x)| = |g(x)|$ is equivalent to the statement $\{f(x)\}^2 = \{g(x)\}^2$. So the given equation is equivalent to

$$(2x-3)^2 = (x+7)^2$$

The answer is $x = 10$ or $-\frac{4}{3}$.

Ex.2 Find all values of a for which the equation $x^2 - ax+1 = 0$ does not have any real root.

Ans: $a \in (-2,2)$

Ex.3 Find the equation whose roots are $\frac{3}{5}$ and $-\frac{7}{9}$.

Ans: $45x^2+8x-21 = 0$

Ex.4 The sum of a number and its reciprocal is 13/6, find all such numbers.

Hint: Let x be the number, then $x + 1/x = 13/6$

Ans: $x = \frac{3}{2}$ or $\frac{2}{3}$

Ex.5 Find two consecutive positive even numbers whose product is 168.

Ans: 12 and 14

Suggested References

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CHAPTER 4

SEQUENCES AND SERIES

1.1 INTRODUCTION

The purpose of this chapter is to introduce the essential features of sequences and series of real numbers to students in order that they can apply the corresponding results in different situations in Mathematics. Not only the concept of sequences and series, but also the corresponding formulae are used in different branches of Mathematics like Calculus, Trigonometry, Mathematical Analysis and Probability etc. Further, these formulae are also used to solve problems in Accountancy, Bank and Industry.

As far as the history of sequences and series is concerned, one has to go back in the period of Vedas. In Yajurved, the sequences 1,3,5,. . and 4,8,12,... have occurred with reference to some rituals In Kalpasutra of Bhadrabahu the result 1+2+4+....+ 8192 = 16383 has been quoted without mentioning the concept of geometric progression. The formulae of section 47 of the [illegible] book were studied first by Aryabhata 1 (see Aryabhatiya Bhashya by Nilkantas). The general formula for the sum to an infinite geometric progression was first given by Vieta in 1590, though the concept of convergence was not mentioned.

The formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

gives the value of π and was proved by Gregory, but there is sufficient evidence that this was known to Madhava (1350 A.D. - 1425 A.D.) of

Iringalakuda of present day Kerala. Among other series known to the Hindus earlier than the ~~Western are the series~~ for sine and cosine functions.

2. CONTENT ANALYSIS

In this section the number of each subsection is in accordance with the text book.

With the help of sufficient number of examples, relationship of n with a_n is to be made clear. In an arithmetic progression, the difference between any two consecutive terms remains same, while in a geometric progression, the ratio of any two consecutive terms remains same. Not only the formula for sum to n terms is to be given but the procedure for the same should also be demonstrated with the help of some sequences. Reason for inability to sum to infinite terms of a G.P., when $|r| \geq 1$, should be made clear. Students should be given such a practice that even if they forget the formulae, they may be able to use the procedure involved in them.

Sequences following certain patterns are more often called progressions. A progression is a particular type of sequence for which a formula for nth term can be given.

e.g. 3,6,12,24,.... is a progression.

Here $t_n = 3 \cdot 2^{n-1}$, $n \in N$. Further, a term 'series' needs to be explained clearly. If all the terms of a sequence are written in the form of a sum, then such a sum is called a series which may or may not be finite according to the situation i.e. the corresponding sequence being finite or infinite.

e.g. series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ is infinite as it corresponds to an infinite sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

4.2 Arithmetic Progression (A.P.)

Problems like Example 4.6 should be taken before Exercise 4.1 and before dealing with the recognition of an A.P. & G.P. etc as well.

4.3 Examples of A.P. And Insertion Of Arithmetic Means

Problems of the type of Example 4.1 should be dealt with after Exercise 4.1 and even after dealing with the easy problems like 1,2,3,...., 2,4,6,.... and 1,3,5,... etc. Students should be encouraged to use the appropriate formula for the sum of an A.P. viz $\frac{n}{2}[2a+(n-1)d]$, $\frac{n}{2}(a+\ell)$ according to the terms given. Further, to fix the idea that $\{a_n+b_n\}$ will be an A.P. if $\{a_n\}$ and $\{b_n\}$ are A.P's. Students may be asked to verify the nature of $\{a_n+b_n\}$, its first term and common difference. e.g.

If $\{a_n\} = \{2,4,6,8,....\}$, $\{b_n\} = \{3,6,9,12,...\}$.

Then $\{a_n+b_n\} = \{5,10,15,20,....\}$.

Its first term, 5, is the sum of the first terms of $\{a_n\}$ and $\{b_n\}$ and the common difference is the sum of their common differences.

The students may also be encouraged to verify the nature of a sequence obtained by

(i) Adding a constant to each term of an A.P.

(ii) Multiplying each term of an A.P. by a constant.

Harmonic progression may be introduced as follows.

If $\{a_n\}$ is an arithmetic progression (A.P.), then $\{\frac{1}{a_n}\}$ is a harmonic progression (H.P.) e.g. $\{2n+1\}$ being an A.P, $\{\frac{1}{2n+1}\}$ will be an H.P. Some easy problems involving harmonic progressions may also be taken. More questions of each type should be given for sufficient practice.

3. LEARNING OUTCOMES

(a) Essential Learning Outcomes For All

After going through this chapter a student should be able to

(i) Recognise an A.P., G.P. and A.G.P.

(ii) Find the common difference d and the common ratio r of given A.P. and G.P. respectively.

(iii) Find a_n where a,d,n (of an A.P.) or a,r,n (of a G.P.) are gi

(iv) Find n when a,d,a_n (of an A.P.) or a,r,a_n (of a. G.P.) are g

(v) Calculate S_n of an A.P. or a G.P.

(vi) Insert given number of arithmetic means or geometric means between two given numbers.

(vii) State whether the sum of an infinite G.P. can be found or no and find sum to infinity, if possible.

(viii) Calculate sum of an infinite A.G.P., if possible

(ix) Sum to n terms of some special sequences.

(b) Learning Outcomes For The Higher Group

After going through this chapter, a student of higher group should be able to

(i) Calculate S_r of an A.P. or a G.P. even without using a formu

(ii) Find the nth term of an H.P. when its first two terms are gi

(iii) Insert given number of H.Ms between two given numbers

(iv) Establish order relation between A.M., G.M. and H.M. of two given positive numbers

(v) Sum to n terms of some special sequences which are not G.P.s but reducible to G.P.s

(vi) [illegible] the problems of application type.

TEACHING STRATEGIES

a) Motivation For the Development of Concepts

Every experienced teacher knows that any mathematical topic can well taught to the students if they are well motivated. Motivation can be given through simple examples.

To start with the topic, different day to day life situations may be presented where we come across. To judge whether the students have recognised the pattern, they may be asked to find successive terms. Some interesting sequences like Fibonacci sequence 1,1,2,3,5,7,... may be explained with the logic behind it.

Sequence $1,2,2^2,\ldots$ may be presented relating it with the story of a king and a chess maker $1+2+2^2+\ldots+2^{63}$ being the total number of grains to be paid to the chess maker as a reward.

Series $2+1+\frac{1}{2}+\frac{1}{4}+\ldots$ may be presented with the story of a rabit who wishes to cross a distance of 4 metres. In his first jump he covers 2 m but in next jumps he covers half of the distance covered in the preceding jump. Now the question is, will he ever be able to reach his destination? If 'yes', in how many jumps?

While introducing S_n, need to find S_n may be explained/displayed. The students will feel motivated. Even to derive a simple formula, it is always better to consider a corresponding example.

Suppose we want to find the A.M. of two numbers a and b, give example of the following type.

Example 1: Find A such that 3, A and 11 are in A.P.

Consider the following example

$$2,1,7,-7,\sqrt{2},8,7,5,1,100$$

This is a finite sequence consisting of 10 elements written in order. One can tell any of the 10 terms even when it is not possible to give a formula for a_n.

(c) <u>Additional Exercises</u>

(i) A person working in an office gets Rs. 2000/- as a basic pay and 50% D.A. per month. The increment per year is Rs. 60/- in the basic pay. He saves 10% of his basic pay in his provident fund account every month. Find the total amount of his contributio towards the provident fund at the end of 5 years.
(Hint: Additional information about D.A. is of no use in the solution of the problem) (Ans: Rs. 12720).

(ii) The odd natural numbers are grouped as follows
1;3,5;7,9,11;13,15,17,19;....
Show that the sum of the numbers in the nth group are n^3.

(iii) Give an example of a sequence which is A.P., G.P. as well as H.P.

(iv) If a,b and c are in G.P., prove that

$$\frac{1}{a+b}+\frac{1}{b+c}=\frac{1}{b}$$

(v) Find the sum of all numbers between 50 and 150 which are divisible by 7. (Ans: 1421)

(vi) Sum to n terms the series
4 + 44 + 444 + ...
(Hint: $\frac{4}{9}\,[9+99+999+\ldots] = \frac{4}{9}\{(10-1)+(10^2-1)+(10^3-1)+\ldots\}$)

(Ans: $\frac{40}{81}(10^n-1)-\frac{4}{9}n$)

(d) Solutions/Hints For Difficult Problems

Exercise 4.1:

Q2. Hint: Find a_2-a_1 and a_3-a_2 to show that

$$a_2-a_1 \neq a_3-a_2$$

Exercise 4.2:

Q3. Hint: $S_6 = 5(S_{12} - S_6)$

$$\Rightarrow 6S_6 = 5S_{12}$$

$$\text{or } 6\left[\frac{6}{2}(2a+5d)\right] = 5\left[\frac{12}{2}(2a+11d)\right]$$

Exercise 4.3:

Q2. Hint: Suppose the numbers are a, ar, ar^2, ar^3

$$ar^2 - a = 9 \quad \text{— (1)} \qquad (1)$$

$$ar - ar^3 = 18 \quad \text{— (2)} \qquad (2)$$

Multiplying (1) by r, we get

$$ar^3 - ar = 9r \quad \text{— (3)} \qquad (3)$$

From (2) and (3) we get

$$9r = -18$$

$$\Rightarrow r = -2$$

Exercise 4.6:

Q1. Hint: $2^2 + 4^2 + \ldots + (2n)^2$

$$= 2^2[1^2 + 2^2 + \ldots + n^2]$$

Miscellaneous Exercises on Chapter 4

Q7. Solution: The sum of the distances covered is given by

$$S = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$$

This is a G.P. with common ratio, $\frac{1}{2} < 1$

$$\therefore S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2 < 3$$

which shows that the insect can never reach the target point.

Q11. Solution:

We have $A = \frac{a+b}{2}$, $G = \sqrt{ab}$ and $H = \frac{2ab}{a+b}$.

Now $A.H. = \frac{a+b}{2} \cdot \frac{2ab}{a+b} = ab = (\sqrt{ab})^2 = G^2$

A, G and H are in G.P.

Further, $A-G = \frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}(a+b-2\sqrt{ab})$

$$= \frac{1}{2}(\sqrt{a}-\sqrt{b})^2 > 0$$

$\therefore \quad A > G$

Again $G-H = \sqrt{ab} - \frac{2ab}{a+b} = \frac{(a+b)\sqrt{ab}-2ab}{a+b}$

$$= \frac{\sqrt{ab}}{a+b}[a+b-2\sqrt{ab}]$$

$$= \frac{\sqrt{ab}}{a+b}(\sqrt{a}-\sqrt{b})^2 > 0$$

Q12. Solution:

$$S_1 = \frac{n(n+1)}{2}, \quad S_3 = S_1^2$$

and $S_2 = \frac{n(n+1)(2n+1)}{6} = \frac{S_1(2n+1)}{3}$

$$9S_2^2 = \frac{9\times S_1^2(2n+1)^2}{9} = S_1^2(2n+1)^2$$

$$= S_3(4n^2+4n+1) = S_3\left[1+\frac{8(n+1)n}{2}\right]$$

$$= S_3(1+8S_1)$$

Q.E.D.

Other Methods of Solving Some Of The Problems

Problem (I): Prove that $1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof: We shall adopt the method of induction discussed in Chapter 6 of the text book

P(1): $1^2 = \frac{1(1+1)(2+1)}{6}$

or $1 = 1$ which is true

P(1) is true

P(r); $1^2+2^2+3^2+\ldots+r^2 = \frac{r(r+1)(2r+1)}{6}$.

Let us assume that it is true

P(r+1): $1^2+2^2+3^2+\ldots+(r+1)^2 = \frac{(r+1)(r+2)(2r+3)}{6}$

LHS: $\underbrace{1^2+2^2+3^2+\ldots+r^2}+(r+1)^2$

$$= \frac{r(r+1)(2r+1)+6(r+1)^2}{6}$$

$$= \frac{(r+1)}{6}[2r^2+r+6r+6]$$

$$= \frac{(r+1)(r+2)(2r+3)}{6} = \text{RHS}$$

This shows that P(r+1) is true whenever P(r) is true.

$\therefore$ By principle of mathematical induction P(n) is true for all $n \in N$.

Problem (II): Derive the formula

$$a+ar+ar^2+\ldots+ar^{n-1} = \frac{a(r^n-1)}{r-1}$$

Solution: We shall prove it also by the method of induction

P(1): $a = \frac{a(r^1-1)}{r-1} = a$ which is true

$\therefore$ P(1) is true.

$P(k)$: $a+ar+ar^2+\ldots+ar^{k-1} = \dfrac{a(r^k-1)}{r-1}$

Let us assume that it is true

$P(k+1)$: $a+ar+ar^2+\ldots+ar^k = \dfrac{a(r^{k+1}-1)}{r-1}$

$$\text{L.H.S.} = \underline{a+ar+ar^2+\ldots+ar^{k-1}}+ar^k$$

$$= \frac{a(r^k-1)}{r-1}+ar^k \quad (\because P(k) \text{ is true})$$

$$= \frac{ar^k - a + ar^{k+1} - ar^k}{r-1}$$

$$= \frac{a(r^{k+1}-1)}{r-1} = \text{R.H.S.}$$

This shows that $P(k+1)$ is true whenever $P(k)$ is true.

$\therefore$ By principle of mathematical induction $P(n)$ is true for all $n \in N$.

(f) Some Challenging Problems

Q(1) The odd natural numbers are grouped as follows

1; 3,5; 7,9,11; 13,15,17,19;

Show that the sum of the numbers in the nth group is n^3.

Solution:

The first terms of each group are

1,3,7,13,21,...

These can be rewritten as

$0\times1+1,\ 1\times2+1,\ 2\times3+1,\ 3\times4+1$,

$\therefore$ The first term of the nth group is

$(n-1)n+1 = n^2-n+1$

Then there are n terms in the nth group.

These terms form an AP with common difference 2

$$\therefore \quad S = \frac{n}{2}[2a + (n-1)d]$$

$$= \frac{n}{2}[2(n^2-n+1) + 2(n-1)]$$

$$= n(n^2 - n+1 + n-1)$$

$$= n^3.$$

Q(ii) Sum the series

$$1+(1+2) + (1+2+3) + (1+2+3+4)+\dots \text{ n terms}$$

Solution:

The nth term of the given series is

$$1+2+3+\dots+n = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$\therefore \quad S_n = \Sigma(\frac{1}{2}n^2 + \frac{1}{2}n)$$

$$= \frac{1}{2}\Sigma n^2 + \frac{1}{2}\Sigma n$$

$$= \frac{1}{2}[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}]$$

$$= \frac{n(n+1)}{12}[2n+1+3] = \frac{n(n+1)(n+2)}{6}$$

Q(iii) If the sum of the first m terms of an A.P. is equal to the sum of the first n terms of the same A.P. then show that the sum of the first (m+n) terms of the A.P. is zero.

Solution:

Let a be the first term and d, the common difference of the A.

Then we have

$$\frac{m}{2}[2a+(m-1)d] = \frac{n}{2}[2a+(n-1)d]$$

$$\therefore \quad 2a(m-n) + d[m(m-1)-n(n-1)] = 0$$

i.e. $(m-n)[2a+(m+n-1)d] = 0$

but $m \neq n$

$\therefore$ $2a(m+n-1)d = 0$

$\therefore$ $\frac{m+n}{2}[2a+(m+n-1)d] = 0$

i.e. sum of the first $(m+n)$ terms of the A.P. is zero.

Q(iv) If pth term of an A.P. is q and q^{th} term is p, then show that $(p+q)^{th}$ term of the A.P. is zero.

<u>Hint:</u> Use the procedure adopted in Q(iii) above.

Q(v) Sum the series

$1\times2 + 2\times3 + 3\times4 + \ldots + n(n+1)$

<u>Hint:</u> n^{th} term $= n^2 + n$

$\therefore$ Sum $= \Sigma n^2 + \Sigma n$ (Ans: $\frac{n(n+1)(n+2)}{3}$)

Q(vi) Sum the series

$\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \ldots$ upto n terms

<u>Solution:</u>

n^{th} term $= \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

1^{st} term $= \frac{1}{1} - \frac{1}{2}$

2^{nd} term $= \frac{1}{2} - \frac{1}{3}$

3^{rd} term $= \frac{1}{3} - \frac{1}{4}$

n^{th} term $= \frac{1}{n} - \frac{1}{n+1}$

$\therefore$ $S_n = 1-\frac{1}{n+1} = \frac{n}{n+1}$

A rotating wheel coming to rest rotates 300 revolutions in the first minute. If in each subsequent minute it rotates two thirds of the rotations in the preceding minute, how many revolutions will the wheel make before coming to rest?

(Ans: 900)

) A gives Rs. 20/- as a donation to a school everyday while B gives 50 p on the first day, Re 1/- on the second day, Rs. 2/- on the third day and so on. At the end of 10 days, whose donation will be more and by how much amount?

(Ans: B; by Rs. 11.50)

CHAPTER TESTS

Oral Test

There can be various types of oral questions related to the topic concerned. A few questions are mentioned here. A teacher can frame his own questions. While framing such questions more stress should be given on concept understanding than on the application of the formulae involved

What is the 5^{th} term of the sequence 21,21,21,...

What is the formula for the n^{th} term of the suquence -3,-3,-3,...

What is the 7th term of the sequence 1,-1,1,-1,... (Ans:1)

What is the common difference of an A.P. whose n^{th} term is $4n-1$

(Ans: 4)

What is the common difference of an A.P. whose n^{th} term is $3n^2-5n$

(Ans: 6)

Give an example of a sequence which is A.P., G.P. and H.P.

Does the sum to infinity for the following G.P. exist? Why?

$1+1.1+(1.1)^2+(1.1)^3+\ldots$

(Ans: no, because $|r| > 1$)

(viii) What is the sum of the series
$1+2+3+\ldots+100$. (Ans: 5050)

(ix) What is the 4^{th} term of the sequence for which m^{th} term is $5m-7$? (Ans: 13)

(x) What is the 2^{nd} term of a sequence whose n^{th} term is $3n^2-4$? (Ans: 8)

(xi) What is the common ratio of a G.P. whose n^{th} term is $5\left(\frac{3}{2}\right)^{n-1}$? (Ans: $\frac{3}{2}$)

(xii) What is the first term and common ratio of a G.P. whose n^{th} term is $\left(\frac{3}{4}\right)^{n+1}$. (Ans: $\frac{9}{16}$; $\frac{3}{4}$)

(xiii) Find if the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is
(A) A.P. (B) G.P. (C) H.P. (D) A.G.P. (E) None of these
(Ans: C)

(xiv) Find if the sequence whose n^{th} term is $3n+4$ is
(A) H.P. (B) G.P. (C) A.P. (D) A.G.P. (E) None of these
(Ans: C)

(b) <u>Written Test</u>

(i) Derive the formula for the n^{th} term of an H.P. whose first two terms are a and b

(Ans: $t_n = \dfrac{1}{\frac{1}{a}+(n-1)\left(\frac{a-b}{ab}\right)}$)

(ii) Insert 3 harmonic means between $\frac{1}{6}$ and $\frac{1}{42}$

(Ans: $\frac{1}{15}, \frac{1}{24}, \frac{1}{33}$)

(iii) Sum the series

$(2\times5) + (3\times6) + (4\times7) + \ldots$ to n terms

(Ans: $\frac{n}{3}(n^2+9n+20)$)

(iii) Sum the series
$(2\times 5)+(3\times 6)+(4\times 7)+\cdots$ to n terms
(Ans: $\frac{n}{3}(n^2+9n+20)$)

(iv) How many terms should be taken so that the sum to that number of terms of a G.P. whose 1st term and common ratio are 1 and 2 respectively, is 2^9-1.

(Ans: 9 terms)

(v) If in an A.P. $S_2 = S_9 = 9$, show that 6^{th} term is zero.

(vi) The angles of a triangle are in A.P. and the greatest angle is 100°. Determine the smallest one.

(Ans: 20°)

(vii) The digits of a pin code number, taken in the order, of a town are in A.P. If the number is divisible by 8, find it.

(Ans: 765432, 123456)

(viii) Sum to infinity the following series

$1+\frac{4}{3}+\frac{7}{9}+\frac{10}{27}+\ldots.$ (Ans: $\frac{15}{4}$)

(ix) The sum of an infinite G.P. is 4. Find the common ratio if the first term is 3.

(Ans: $\frac{1}{4}$)

(x) The ratio of the 5^{th} term of a G.P. to its 2^{nd} term is $\frac{27}{8}$. Determine the ratio of 4^{th} term to the 2^{nd} term.

(Ans: $\frac{9}{4}$)

6. <u>ADDITIONAL READING MATERIAL FOR ENRICHMENT</u>

(i) College Algebra by Raymond A.Bornett

(ii) Higher Algebra by Barnard and Child

(iii) Higher Algebra by Hall and Knight

(iv) Algobra by Fuler

(v) Exploring Modern Mathematics by K.V. Rao

(vi) High School Mathematics Part 1 by G.N. Yakovlov, Mir Publishers, Moscow.

CHAPTER 5

PERMUTATIONS AND COMBINATIONS

1. INTRODUCTION

There are certain situations where one is interested in knowing the number of ways a thing or a work can be done. Sometimes one also desires to obtain this number, not by actually counting the different ways. In dealing with such problems one has to arrange different or identical things or has to find the number of groups without bothering the arrangement of things in each group. All such and other related problems in which the question of counting occurs are dealt with, in this chapter.

Permutations and combinations are used in dealing with Binomial Theorem, which is discussed in the next chapter. Further, the idea of permutations and combinations alongwith the corresponding formulae, is used in dealing with problems of Probability.

Combinatorial Analysis, a well known branch of mathematics, which deals with descrete objects and configurations obtained by them, has its origin in the theory of permutations and combinations. There is another branch of mathematics called 'Graph Theory' which is closely related to Combinatorial Analysis. These two branches have their applications in Computer Science, Assignment Problems and Number Theory.

Historically, the problems related to combinations and permutations arose while solving descrete probability problems (A non-continuous variable is known as a descrete variable). Pascal, Euler, Lagrange, Laplace, Bernoulli, Leibnitz were pioneers in this field and solved many problems on probability by using combinatorial

techniques. Among Indian mathematicians, Bhaskaracharya (Eighth Century) dealt in some problems related to permutations and combinations in his work.

No prerequisite knowledge is required to understand this chapter.

2: CONTENT ANALYSIS

In this section the numbers of each sub-section are in accordance with the text book.

5.2: A student may be able to understand the fundamental principle of counting (also known as principle of multiplication) in a better way if the same may be stated as follows.

"If one thing or one operation can be done or performed in m ways and to each way of doing the first thing (or the first operation), the second thing can be done in n ways, then the number of doing the two things simultaneously is m.n".

Along with this principle, the following principle may also be stated, since this is also equally important.

Principle of Addition: If a thing can be done in m ways, and also by n ways, all the ways being different, then the thing can be done in (m+n) ways.

Sometimes one has to use both the principles in solving some problems depending on the nature of the problem and the cases arise therein. See for example Exercise 6 on page 99 of the Text Book.

5.4 to 5.7: In these sections the concepts of Permutations and combinations are dealt in, along with illustrative examples. These sections also deal with certain theorems. The proofs of these

theorems are very short and need more elaboration. The more elaborate proofs are given in section 4 of this chapter.

Teachers may go through the proofs and form their own proofs written in an easier language.

The difference between Permutation and Combination can be illustrated and explained by giving a stress on order of the things in an arrangement. Further while dealing with permutations and combinations, the following problems should be discussed and explained seperately.

(i) Problems dealing with permutations and combinations of DIFFERENT Things.

(ii) Problems dealing with permutations and combinations for which, some or all the things are identical.

There is a possibility of using a formula related to different things for solving a problem in which some things are identical. A repeated warning is to be given to the students regarding this point.

3. LEARNING OUTCOMES

(a) Essential Learning Outcomes For All Students:

After learning this chapter each student is supposed to

(i) recognise the fundamental principle of counting and its use in solving simple problems,

(ii) recognise the factorial notation for whole numbers and its use and purpose

(iii) recognise the meaning of the terms "permutation" and "combination", and the difference between them.

(iv) remember and use different variety of formulae related to permutations and combinations, in solving simple related problems

(v) analyse and solve simple practical problems.

(b) <u>Learning Outcomes For Students Of Higher Group</u>

A student of higher group is expected to,

(i) prove the theorems on permutation and combination,

(ii) apply the formulae on permutation and combinations, to solve harder problems.

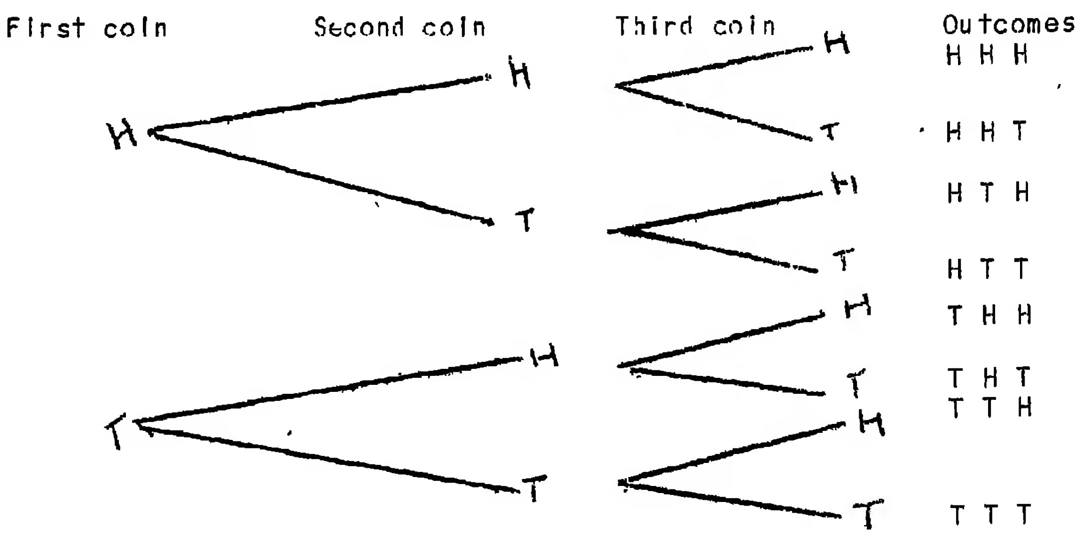
4. <u>TEACHING STRATEGIES</u>

(a) <u>Motivation</u>: In order to motivate the concept of permutation, the following simple example can be discussed with the help of a figure.

"How many possible outcomes can be obtained if three coins are tossed?"

<u>Solution</u>: Let letter H represent the occurrence of Head and T represent the occurrence of Tail.

Then the outcomes are shown in the following figure.



The problems discussed is also equivalent to find all possible three letter words using the letters T and H.

For further motivation of the concepts of permutation and combination the following examples will also be useful.

(I) A student has 5 shirts and 3 pants. How many different dresses can be formed, each dress consisting of one shirt and one pant.

(II) A person desires to travel from Bombay to Delhi via Ahmedabad. He can travel from Bombay to Ahmedabad either by a Rail or by a bus; while from Ahmedabad to Delhi there are three ways of travelling (i) by train (ii) by bus (iii) by air.

In how many ways can the person perform the journey from Bombay to Delhi. Illustrate the number of ways by drawing a figure.

(III) A father has asked his son to bring one pair of hand gloves out of 5 pairs kept in a room. When the son reached inside the room the lights were off. (It was a night time). The father was in hurry as he wanted to travel a nearby village. How many minimum number of hand gloves the son should take and give to his father so that the father should be able to use at least one pair of hand gloves.

A teacher after discussing such problems alongwith the solutions thereof, may ask the students to tell such similar problems.

A problem of the following type may also be considered.

(IV) Write down all English words that can be formed by using letters of the word "MATHEMATICS" partial solution is HE, I, AT, HIT, MAT, HIS etc.

In order to clarify the difference between permutation and combination, following examples may be discussed.

(v) Arrange the symbols ॐ, 卐 and  in different ways along a line.

(vi) There are five coins in my pocket. These are 5 ps, 10, ps, 20 ps, 25 ps and 50 ps respectively. I want to pick up any three coins at a time without looking into the pocket and without bothering which three coins are going to be picked up. In how many ways this can be done? In which way, what is the amount I shall get?

This last example also illustrates and motivates the formula.

$$C(n,r) = C(n,n-r)$$

Here we have $C(5,3) = C(5,2) = 10$

<u>Remark:</u> A teacher is welcome if he or she explains some of the above cited examples with the help of figures and diagrams. For instance the 2nd example (ii) may be illustrated through the following figure

Bombay 

The different possible ways of travelling are as follows:

(i) and (Train, Train)

(ii) and (Train, Air)

(iii) and (Train, Bus)

(iv) and (Bus, Train)

(v) and (Bus, Air)

(vi) and (Bus, Bus)

<u>Misconceptions and Common Errors:</u>

Sometimes a student does not give due importance to the concept of order and finds difficulty in solving the problems on permutation and combination.

Secondly a student does not pay necessary attention to the case of permutation in which the things are repeated. The following problem illustrates the second point.

<u>Example:</u> Find the number of three digit positive integers that can be formed by using the digits 5,6,1 and 9.

Here an average student takes P(4,3) and gets a wrong answer 24. Though it is not mentioned in the problem explicitly that a digit may be repeated, it is to be taken for granted, as it is not otherwise stated.

The correct answer is 4x4x4 = 64.
Further, if one of the four given digits happens to be zero, the answer is not 64; but it is

$$3x4x4 = 48$$

Another common error a student may commit is not to consider the different cases that arise in a particular problem. For example consider the following problem.

There are four apples to be distributed among 3 students A,B,C so that A should get atleast two.

<u>Solution:</u> There are three cases to be considered.

<u>Case (i) :</u> A gets 2 apples

<u>Case (ii):</u> A gets 3 apples

<u>Case (iii):</u> A gets 4 apples

If a student considers only the first case he gets a wrong answer

Solutions/Hints to Difficult Problems of The Text Book

Exercise 5.1:

Q9. For a set of five true-or-false questions, no student has written all the answers correct, and no two students have given the same sequence of answers. What is the maximum number of students in the class, for this to be possible?

Solution: The two symbols T and F can be written at 5 different places in 2^5 ways i.e. 32 ways. Among these only one is related to the all correct answers of 5 questions. Hence the remaining 31 permutations are all different.

Hence the maximum number of students is 31.

Exercise 5.2:

Q9. Prove the inequality

$(n!)^2 \leq n^2.n! \leq (2n!)$ for all positive integers n.

Hint: Apply Induction Principle.

Q13. Prove that 33! is divisible by 2^{15}. What is the largest integer n such that 33! is divisible by 2^n?

Proof: There are at least 16 even factors (2,4,6,..., 16) of 33!. Hence 2^{15} surely divides 33!. Let us consider the number of times the factor 2 may occur in the sequence.

2,4,6,8,10,....,32.

It amounts to 1+2+1+3+1+2+1+4+1+2+1+3+1+2+1+5 = 31. Hence n = 31.

Exercise 5.5:

Q8. If there are six periods in each working day of a school, in how many ways can one arrange 5 subjects such that each subject is allowed at least one period.

Solution: If any one of the subjects occurs twice then there are $\frac{6!}{2!}$ ways of arranging periods. But there are 5 subjects. Hence the desired number of ways $= \frac{6!}{2!} \times 5 = 1800$. The answer given in the book is wrong.

010. Find a formula for the number of permutations of n different things taken r at a time, such that two specified things occur together.

Solution: Considering two specified things as one object we get $P(n-1,r-1)$ ways. But two specified things can be interchanged. Hence the desired number of ways is $2P(n-1,r-1)$.

Exercise 5.7:

05. From a class of 25 students, 10 are to be chosen for an excursi party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can they chosen?

Solution: Case (i) When 3 students join the excursion party. Number of ways of selecting 7 more students from the remaining 22 is equal to $C(22,7)$.

Case (ii): When 3 students do not join the excursion party. Number of ways of selecting 10 students out of 22 = $C(22,10)$. Hence required number of ways = $C(22,7) + C(22,10)$.

014. Prove that $n!+1$ is not divisible by any number between 2 and n.

Proof: We know that any integer z between 2 and n divides $n!$; but none of these integers divides 1, and hence z cannot divide $n!+1$ (Why?).

Exercise 5.4:

Q8. In an examination hall, there are four rows of chairs. Each row has 8 chairs, one behind the other. There are two classes sitting for the examination, with 16 students in each class. It is desired that in each row, all students belong to the same class and that no two adjacent rows are alloted to the same class. In how many ways can these 32 students be seated.

Solution: Rows are to be alloted to both the classes as I, III, II, IV. Number of ways of selecting class for each row is 2. Number of ways of seating 16 students in I, III (or II, IV) is 16!. So the required number of ways = 2x16!x16! i.e. $2\times(16!)^2$.

Exercise 5.7:

Ex. 6: From a class of 12 boys and 10 girls 10 students are to be chosen for a competition, including atleast 4 boys and 4 girls. The two girls who won the prizes last year should be included. In how many ways can the selection be made?

Hint: If two girls are already decided to be included then

No. of Boys	No. of girls
4	6
5	5
6	4

Ans is

$$C(12,4).C(8,4) + C(12,5).C(8,3) + C(12,6) \times C(8,2)$$

Each time we have to select girls out of 8 and not 10.

Exercis 5.8:

We shall give the proof of one identity in this exercise (i.e. Exercise 5). The proofs of the remaining are similar.

Ex.5: Prove that $r.C(n,r) = n.C(n-1,r-1)$.

Proof: $$\text{L.H.S.} = \frac{r.n!}{r!(n-r)!} = \frac{n!}{(r-1)!(n-r)!}$$

$$= \frac{n.(n-1)!}{(r-1)!(n-1-r+1)!} = n.C(n-1,r-1)$$

$$= \text{R.H.S.}$$

Additional Exercises:

(1) There are five speakers A, B, C, D and E. In how many ways can they speak if B speaks after A (not necessarily just after A).

Solution: Considering A and B as two identical objects as 0 and 0, we have to consider the permutations of 5 objects C, D, E, 0, 0. In each permutation replace the first 0 by A and the second 0 by B. We get the answer $5!/2! = 60$.

(2) In a hexagon how many diagonals are there?

Solution: Number of ways of selecting 2 vertices out of 6 is $C(6,2)$. But 6 sides are not to be included as these are not diagonals. Hence the required number of diagonals $= C(6,2) - 6 = 9$.

General Formula: The above problem can be generalised to a polygon of n sides and we get the answer $C(n,2)-n = \frac{n(n-3)}{2}$. The argument of proving the result is similar to the one used in (2) above.

Some Challenging Problems:

(1) In how many ways n identical objects can be distributed among r different persons?

This problem is equivalent to each of the following two problems.

Problem 1: In how many ways a positive integer n can be written in the form of a sum of r terms, each term being an integer ≥ 1.

Problem 2: In how many ways n identical objects can be placed in r different cells?

Solution: We shall concentrate our attention on Problem 1. In the adjacent figure the problem is illustrated when n = 3 and r = 3. The proofs for the general case is similar.

Placing 3 objects (denoted by the letter 0 in the figure) in 3 cells (two vertical lines forms one cell) is equivalent to finding the ways by which 4 vertical lines (3+1) and 3 objects are to be arranged among themselves.

Cell 1	Cell 2	Cell 3
0 0 0	. . .	. . .
. . .	0 0 0	. . .
. . .	. . .	0 0 0
0 0	0	. . .
0 0	. . .	0
0	0 0	. . .
0	. . .	0 0
. . .	0	0 0
. . .	0 0	0
0	0	0

But out of 4 vertical lines, 2 lines have their positions fixed. Hence we have only 2+3 = 5 total things out of which 2 are of one kind and 3 of the other kind. Hence the total number of required ways is equal to $\frac{(2+3)!}{2!\ 3!} = 10$. Clearly the general formula for the number of required ways is

$$\text{No. of ways} = \frac{(n+r+1-2)!}{n! \times (r-1)!}$$

$$= \frac{(n+r-1)!}{n!(r-1)!}$$

or $\quad$ No. of ways = $C(n+r-1,\ r-1)$.

Remark: As a particular case of the above problem.In the following one. We shall put it in only one form.

Find the number of ways of distributing n-identical objects among r different persons so that each person should get atleast one object ($n \geq r$).

Hint: After giving one object to each of the r persons, we are left with (n-r) objects, which are to be distributed according to the problem 1 above.

(Ans: C(n-1,r-1))

Ex. 2: Find the number of ways of distributing 8 applies among 3 girl so that each girl should get atleast one apple.

(Ans: 21)

Hint: n = 8, r = 3 and C(8-1,3-1) = 21

Ex. 3: In how many ways the playing cards can be distributed among 4 partners, playing the game of Bridge? (Cards are distributed one by on

Hint: There are 52 cards. Distribution takes place one after the other. Hence the problem is exactly equivalent to finding the number of permutations of 52 different things.

(Ans: 52!)

Remark: In this problem the answer is independent of the number of partners.

Oral and Written Tests:

(a) Oral Test:

(1) How many permutations of the letters A, B, C, D are there with A always in the 3rd place?

(2) In how many ways the four members A, B, C, D can be placed around a round table with A and B always together?

(3) Write the product 5.7.8.6 using factorial notation.

(4) Can we give a meaning to C(n,r) and P(n,r) for r > n? Explain.

(5) Among the numbers C(6,1), C(6,2), C(6,3), C(6,4), C(6,5), C(6,6), which is the greatest?

(b) Written Test:

(1) From 5 consonants and 3 vowels, 3 consonants and 2 vowels are chosen to form 5 letter words. How many such words may be formed?

(Ans: $5C_3 \times 3C_2 \times 5!$).

(2) In order to give a signal to a ship, 5 flags of 5 different colours are used. If there are 8 different flags of 8 different colours in how many ways the signals can be made?

(Ans: $8P_5$)

(3) Determine different values of n and r so that P(n,r) = 120

(Ans: 6,3; 5,4 and 5,5).

Additional Reading Material:

The following books may be found to be useful for the teachers.

(1) Finite Mathematics by Kemny, Thompson and Snell.

(2) Higher Algebra by Barnard and Child.

(3) Higher Algebra by Hall and Knight.

(4) Dolorami Mathematical Expositions

a) Mathematical games I by Ross Housberger

b) Mathematical games II by Ross Housberger

c) Mathematical games III by Ross Housberger

d) Mathematical morsels · by Ross Housberger

e) Mathematical plums by Ross Housberger

(Published by Mathematical Association of America)

The books (a) to (e) above contain a variety of combinatorial problems, besides the problems in Geometry, Number Theory. There are some problems which have been appeared in Mathematics Olympiads tests.

(5) Exploring Modern Mathematics by K.V. Rao published by United Publishing House in 1975.

CHAPTER 6

BINOMIAL THEOREM AND MATHEMATICAL INDUCTION

1. Introduction

The principle of Mathematical Induction is one of the most powerful methods of generalising statements for all positive integers. This principle can be used to prove certain results occuring in Chapter 5It is also used in proving the well known Binomial Theorem for integral index. which is a part of this chapter.

This principle of the concepts and methods of Mathematics, proof bymathematical induction, is not the invention of a particular individual. In some works of several early mathematicians, the method of induction is involved implicitly. The principle of mathematical induction was known to pythagoreans (sixth century B.C). It is found implicitly in Canepanu's proof of the irrationality of the golden ratio $\frac{\sqrt{5}+1}{2}$,

The discovery of the inference by mathematical inducation in the modern sense is ascribed to Francesco Maurolica who was the first to make a fairly explicit use of the method in a work published in 1575. However, it was not until the seventeenth century that satisfactory formulations of the method are found in the work of Pierre de fermat (1601 - 1665) and Blaise Pascal (1623-1662).

As far as the Binomial Theorem is concerned, it is a powerful tool in obtaining approximations of desired accuracy, where powers of a binomial expression are involved. The expansion of some power of a binomial could be used in solving

certain problems of divisibility, to derive Newton Gregory formula in Numerical Analysis, to solve problems on probability. to evaluate $\lim_{x \to a} \frac{x^n - a^n}{x - a}$ in calculus and in so many other mathematical situations.

Expansion of $(a+b)^n$ for small positive integral values of n were known earlier, but it is difficult to name the identifier. Among the Arabs, Umar Khayyam (about 1130 A.D.) claimed to have discovered the law of expansion of $(a+b)^n$ Al-Zanjani (1262) used it for n=7. It was Pascal's work (1665) which made Europe familier with it. The generalisation of Binomial Theorem for negative integral and fractional index is due to Isac Newton. Maclaurin in 1742 and Euler in 1774 proved the theorem for rational index. In 1826 N.H. Abel, published the first general proof for any complex exponent.

2. <u>Content Analysis</u>

In this section, the following aspects dominate the content.

A general statement needs to be proved for the reason that any number of individual verifications may tend to be false after few more steps.

Notation P(n) is used to denote a statement (over natural numbers). It may be true for all natural numbers or only for a limited number of values.

Given a statement P(n), it will be true for all natural numbers n if and only if

i) P(1) is true; and

ii) P(k) is true $\Longrightarrow$ P(k+1) is true

The text gives several solved examples. But number of questions in the text book is not sufficient. Additional questions may be given to the students for practice.

Detailed explanation with sufficient number of examples and questions on Binomial Theorem is given in the text book. The way in which the general form of binomial theorem significantly varies from the earlier form for a positive integral index is emphasised by listing essential differences between the two forms. No part of the content needs to be further elaborated.

3. Learning Outcomes

Essential learning Outcomes for all

After going through this chapter, student must be able to

i) appreciate the fact that verification of a statement for number of particular cases does not allow us to generalise the statement,

ii) identify the difference between verification and proof.

iii) explain the meaning of the notation P(n).

iv) expalin P(n) for different values of n.

v) apply the principle of Mathematical Induction to prove certain statements.

vi) conclude that violation of any one of the two verifications involved in the principle of mathematical induction leads to invalid generalisation.

vii) expand small positive integral powers of (a+b) by actual multiplication i.e., by distributive law.

viii) prove Binomial Theorem using principle of mathematical Induction.

ix) draw Pascal's triangle for different Values of index n in order to calculate the various binomial coefficients.

x) Calculate the binomial co-efficients with the help of combinations.

xi) apply the Binomial theorem in simple cases.

xii) produce the general statement of the Binomial theorem for rational index.

xiii) prove simple identities using Binomial theorem.

xiv) calculate the approximate values of certain quantities like $\sqrt{99}, 3\sqrt{65}, (15)^{-\frac{1}{4}}$

Learning Outcomes for higher group:

A student should be able to

i) apply the Principle of Mathematical Induction to prove certain statements, which could have been proved by some other method.

ii) appreciate the beauty and utility of the principle of Mathematical Induction as a tool.

iii)? recognise the condition on x as for as the validity of the expansion of $(1+x)^m$ is concerned, where x and m are real numbers.

iv) derive particular cases of expansion from the corresponding general statement.

v) prove difficult identities using Binomial theorem.

4. <u>Teaching Strategies:</u>

<u>Motivation for the development of concepts</u>

To start with Principle of Mathematical Induction, teachers should try to present statements which seems obviously true, but are not true in general e.g. 4, 24, 44, are divisible by 4, so can we say that any number having 4 as its unit's digit is divisible by 4 ? Actually the statement is not true, as 14 is not divisible by 4, even though it's unit's digit is 4.

Sometimes we are not able to find an example by which any statement can be disproved. Even sufficiently large number of examples supporting our statement do not allow us to generalise the statement.

Another statement may be of the type : one more than square of any odd number is even.

It can be easily shown that $1^2 + 1 = 2$ is even.

Here truth of p(k) means that one more than square of k^{th} odd number is even.

If we prove that truth of p(k) implies truth of p(k+1)
then truth of p(1) implies truth of p(2)
truth of p(2) implies truth of p(3)
.......and so on.

This process can continue indefinitely, which implies that statement is true in general.

Before introducing Binomial theorem the teacher is suggested to present a problem like 99^5, which involve long calculations, if actually calculated. After going through clumsy calculations [illegible]

Misconceptions/Common Errors

Sometimes students take it for granted that P(n) will be true for n=1, but this is one of the necessary conditions if we wish to prove any statement for n=1, 2, 3.............

e.g. $P(n)$: n^2-n+1 is divisible by 3

then $P(1)$: $1^2-1+1 = 1$ is divisible by 3.

But this is not true.

Therefore we can't say that n^2-n+1 is divisible by 3. Sometimes assumption of truth of p(k) leads to truth of p(k+1) but p(1) is not true, so truth of p(1) must be checked.

Another common error committed by the students is with any statement related to even numbers or odd numbers. e.g.

$P(n)$: $1+3+5+\ldots+$ is n^2

or $P(n)$: sum of first n odd numbers is n^2.

If question is given in the 2nd form, students forget to take numbers in the form 2n+1. and after assuming $1+3+\ldots+(2k-1) = k^2$ they try to prove $1+3+\ldots(2k-1) + (2k) = (k+1)^2$ similar is the case with any problem related to even numbers.

Sometimes a statement is to be proved for $n \geq 2$. Then in place of p(1) we are to verify the truth of p(2).

While expanding bimomial expression $(2x-y)^5$ students forget to take negative sign with y. They write all the terms with positive sign. But every term with odd power of y will be negative and every term with even power of y will be positive.

Sometimes while expanding $(1+x)^m$ where m is any rational number, students forget to check $|x|<1$. If we have $|x| \geqslant 1$ then we won't be able to sum up the infinite series which is the expansion of $(1+x)^m$.

Sometimes we wish to find the coefficient of certain power of x say 3rd power in the expression $\left(\frac{(1+\frac{3}{4}x)^{-4}(16-3x)^{\frac{1}{2}}}{(8+x)^{\frac{2}{3}}}\right)$. But students miss one or the other term involving third power of x. All the combination from the three expansions must be considered. Here there will be ten such combinations.

Additional exercises to supplement the text-book

Use method of Principle of Mathematical Induction to prove the following Q1 to Q6.

1. If n is an odd integer greater than 1 then (n^2-1) is divisible by 8.

2. $a+(a+d)+(a+2d)+\ldots\ldots+(a+(n-1)d) = \frac{n}{2}(2a+(n-1)d)$

3. $a+ar+ar^2+\ldots\ldots+ar^{n-1} = \frac{a(r^n-1)}{r-1}$

4. $1^3+2^3+3^3+\ldots\ldots\ldots+n^3 = (1+2+3+\ldots\ldots+n)^2$ or $\frac{1}{4}n^2(n+1)^2$

5. $\frac{1}{3.7}+\frac{1}{7.11}+\ldots\ldots$ upto n terms $= \frac{n}{3(4n+3)}$

   (Hint – nth term $= \frac{1}{(4n-1)(4n+3)}$

6. $2^n > n$ for all n

7. Find cubes of (i) 99 (ii) 101 (iii) 48 by using Binomial theorem.

   Ans (970299, 1030301, 110592)

8. Expand the following expressions by using Binomial theorem.

i) $(x+\frac{1}{x})^5 - (x-\frac{1}{x})^5$ Ans $(10x^3+\frac{20}{x}+\frac{2}{x^5})$

iii) $(1+x+x^2)^6$ Ans $(x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+126x^7+141x^6$
$+126x^5+90x^4+50x^3+21x^2+6x+1)$

iv) $(\sqrt{2}+\frac{1}{\sqrt{2}})^7$ Ans $\left(\frac{2187\sqrt{2}}{16}\right)$

9. Find the Coefficient of x^6 in the expansion of $(2x^3+\frac{1}{x})^5$

(Ans 80)

Solutions/Hints for difficult problems

Exercise 6.1

Q5. Hint -

$$2^{3(k+1)}-1 = 2^{3k+3}-1$$
$$= 8.2^{3k}-1$$
$$= 8(2^{3k}-1)+8-1$$
$$= 8(2^{3k}-1)+7$$

Exercise 6.2

Q2. Hint -

This question is same as proving that $3^{2n}-1$ is divisible by $8\ \forall n$.

Q8. Hint - with the assumption that k^3+3k^2+5k+3 is divisible by 3.

$(k+1)^3+3(k+1)^2+5(k+1)+3$

$= k^3+3k^2+3k+1+3k^2+6k+3+5k+5+3$

$= (k^3+3k^2+5k+3)+3k^2+9k+9$

$= (k^3+3k^2+5k+3)+3(k^2+3k+3)$

Q9. Hint:

$$1^2 + 2^2 + \text{------} + k^2 + (k+1)^2$$

$$= \frac{1}{6}k\ (k+1)\ (2k+1) + (k+1)^2$$

$$= \frac{(k+1)}{6}\left[k(2k+1)+6(k+1)\right]$$

$$= \frac{(k+1)}{6}\left[(2k^2 + k + 5k + 6)\right]$$

$$= \frac{(k+1)}{6}\left[(2k^2 + 3k + 4k + 6)\right]$$

$$= \frac{(k+1)}{6}\left[(2k+3)\ (k+2)\right]$$

$$= \frac{(k+1)\ (k+2)\ 2k+3)}{6}$$

<u>Exercise - 6.3</u>

Q11. <u>Solution</u> - Let t_{r-1}, trh t_{r+1} denote the $(r-1)^{th}$, rth and $(r+1)^{th}$ terms respectively in the expansion of $(1+a)^n$. Then we have

$$\frac{\text{Coeff of } t_r}{\text{Coeff of } t_{r-1}} = \frac{7}{1}, \quad \frac{\text{Coeff of } T_r+1}{\text{Coeff of } t_r} = \frac{42}{7}$$

$$\therefore \quad \frac{{}^nc_{r-1}}{{}^nc_{r-2}} = \frac{7}{1} \text{ and } \frac{{}^nc_r}{{}^nc_{r-1}} = \frac{6}{1}$$

$$\therefore \quad \frac{n-r+2}{r-1} = 7 \qquad \text{and} \qquad \frac{n-r+1}{r} = 6$$

Solving these two equations we get n = 55

Q5. Hint

$9^{k+2} - 8(k+1)-9$

$9^{k+1}\, 9 - 8k - 8 - 9$

$9(9^{k+1} - 8k - 9) + 72k + 81 - 8k - 8-9$

$9(9^{k+1} - 8k - 9) + 64k + 64$

$9\ (9^{k+1} - 8k - 9) + 64\ (k+1)$

Q7. Hint

$4^n = (1+3)^n$

$= C(n,0)3^0 + C(n,1)3^1 + C(n,2)3^2 + \text{--------} + C(n,n)3^n$

$$= \sum_{r=0}^{n} C(n,r)3^r$$

Q10. Solution

We want to prove that

$$\frac{(C_0 + C_1)\ (C_1+C_2)\text{------}(C_{n-1} + C_n)}{C_0 C_1 \text{------} C_{n-1}} = \frac{(n+1)^n}{n!} \qquad \text{(I)}$$

Now L.H.S $= (1+\frac{C_1}{C_0})\ (1+\frac{C_2}{C_1})\ (1+\frac{C_3}{C_2}) - (1+\frac{C_n}{C_{n-1}})$

But $\frac{C_r}{C_{r-1}} = \frac{n-r+1}{r}$, $r = 1,2 \ldots\ldots n.$

$\therefore$ L.H.S. of I $= (1+ \frac{n}{1})\ (1+\frac{n-1}{2})\ (1+\frac{n-2}{3}) \ldots\ldots\ldots (1+\frac{1}{n})$

$= (\frac{1+n}{1})\ (\frac{2+n-1}{2})\ (\frac{3+n-2}{3}) \ldots\ldots\ldots (\frac{n+1}{n})$

$= \frac{(1+n)^n}{n!} =$ R.H.S. of I

Q.12 We shall prove the following general result of which the given problem forms a particular case

General Result: Prove that

Proof

$(1+x)^n = C_0+C_1x + C_2x^2+............+C_nx^n$

$(x+1)^n = C_0x^n + C_1x^{n-1} + ...+C_n$

multiplying these identities, we get

$(1+x)^{2n} = (C_0+C_1x+C_2x^2+...............+C_nx^n) \times (C_0x^n+C_1x^{n+1}+....+C_n)$

Equalting the coeff. of x^{n-r} on both the sides,

We get $C(2n, n-r) = C_0C_r + C_1C_{r+1}+.......+C_{n-r}C_n$

$\frac{(2n)!}{(n-r)!(n+r)!} = C_0C_r+C_1C_{r+1} + ------+C_{n-r}C_n$

Now putting r=1 in this result

$\frac{(2n)!}{(n-1)!\ (n+1)!} = C_0C_1+C_1C_2+......+ C_{n-1}C_n$

Q.15 Hint

$(1+x)^n (1-x)^n = (1-x^2)^n$

Compare the coeff. of x^4 on both the sides.

Exercise 6.5

Q4. Solution

Term involving x^r in $(1-4x)^{-\frac{1}{2}}$.

$t_{r+1} = \frac{(-\frac{1}{2})(-\frac{1}{2}-1)..............(-\frac{1}{2}-(r-1))}{1 \times 2 \times 3 \times .... \times r} (4x)^r$

$\therefore$ Coeff. of $x^r = \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})......(-\frac{2r-1}{2})(-4)^r}{r!}$

$= \frac{(-1)^r (-1)^r 4^r (1 \times 3 \times 5 \times ...(2r-1))}{2^r\ r!}$

$= \frac{2^r(1 \times 2 \times 3 \times 4 \times .....(2r))}{(2 \times 4 \times ....2r)r!}$

Q8 **Solution** Take $p=q+h$

$$\text{L.H.S.} = \frac{(n+1)\ (q+h)+(n-1)q}{(n-1)\ (q+h)\ +(n+1)q} = \frac{2nq\ +(n+1)h}{2nq\ +(n-1)h}$$

$$= \frac{2nq\left[1+\frac{(n+1)h}{2nq}\right]}{2nq\left[1+\frac{(n-1)h}{2nq}\right]} = \left(1+\frac{(n+1)h}{2nq}\right)\left(1+\frac{(n-1)h}{2nq}\right)^{-1}$$

$$= \left[1+\frac{(n+1)h}{2nq}\right]\left[1-\frac{(n-1)h}{2nq}\right] \qquad \text{(as h is small)}$$

$$= 1+\frac{[n+1-(n-1)]}{2nq}\,h \qquad \text{(as h is small)}$$

$$= 1+\frac{2h}{2nq} = 1+\frac{h}{nq}$$

$$\text{R.H.S.} = \left(\frac{q+h}{q}\right)^{\frac{1}{n}} = \left(1+\frac{h}{q}\right)^{\frac{1}{n}} = 1+\frac{1}{n}\frac{h}{q} \ \text{(as h is small)}$$

Q10 **Hint**

$$(a-bx)^{-2} = a^{-2}\left(1-\frac{b}{a}x\right)^{-2}$$

$$= a^{-2}\left(1+2\left(\frac{b}{a}x\right)+3\left(\frac{b}{a}x\right)^2+\dots\dots\dots\right)$$

if all the coeff. are positive then $\frac{2b}{a} > 0$

$\Rightarrow$ b and a are of the same sign.

Q12. **Hint**

$$(1-x)^m = 1+m\,(-x) + \frac{m(m-1)}{1\times 2}(-x)^2+\dots\dots\dots\dots$$

coeff. of $x^2 = \frac{m(m-1)}{1\times 2}$

hence $\frac{m(m-1)}{2} = 3$

or $m^2-m-6=0$

$\Rightarrow$ $m= -2,\ 3$.

Miscellaneous Exercises 6.6

Q3. **Hint**

$$(1+x)^{k+1}$$
$$= (1+x)^k (1+x)$$
$$\geqslant (1+kx)(1+x)$$
$$= 1+x+kx+kx^2$$
$$\geqslant 1+(k+1)x \qquad (\text{as } k>0,\ x^2>0)$$

Q6. i) Hint

By assumption

$$(k+3)^2 \geqslant (2k+7)$$

Consider $[(k+1)+3]^2 = [(k+3)+1]^2$

$$= (k+3)^2 + 1^2 + 2 \times 1 \times (k+3)$$
$$\geqslant 2k+7+1+2k+6$$
$$= 2k+9+2k+5$$
$$\geqslant 2(k+1)+7 \qquad (\because 2k+5 \geqslant 0)$$

Hence: $(n+3)^2 \geqslant (2n+7)$

ii) **Hint** - By assumption

$$(k+3)^2 \leqslant 2^{k+3}$$

Consider $(k+4)^2 = (k+3)^2 + 1^2 + 2(k+3)$

$$= (k+3)^2 + 2k+7$$
$$\leqslant (k+3)^2 + (k+3)^2 \qquad [\text{By (i)}]$$
$$\leqslant 2^{k+3} + 2^{k+3}$$
$$= 2^{k+4}$$

By assumption

hence $(n+3)^2 \leqslant 2^{n+3}$

<u>Addtional applications</u>:- There are a number of applications of induction principle in various branches of mathematics. We shall quote here two applications, one taken from Trigonometry and the other from the theory of numbers.

(1) For any positive integer n prove that $(\cos\theta + i\sin\theta)^n =$

$(\cos n\theta - i\sin n\theta)$ ---- (P(n)) where $i = \sqrt{-1}$

<u>Proof</u>:- Clearly the result (P(n)) is true for n=1.

Let us assume that P(n) is true for n=r i.e. let

$(\cos\theta = i\sin\theta)^r = (\cos r\theta + i\sin r\theta)$

multiplying both sides by $(\cos\theta + i\sin\theta)$ we get

$(\cos\theta + i\sin\theta)^{r+1} = (\cos r\theta \cos\theta - \sin r\theta \sin\theta)$

$+ i(\sin r\theta \cos\theta + \cos r\theta \sin\theta)$

using the formulae

$\cos A \cos B - \sin A \sin B = \cos(A+B)$

and $\sin A . \cos B + \cos A \sin B = \sin(A+B)$

we get

$(\cos\theta + i\sin\theta)^{r+1} = \cos(r+1)\theta + i\sin(r+1)\theta$

This shows that P(r+1) holds.

Thus P(r) implies P(r+1).

Hence by the principle of mathematical induction we conclude that P(n) holds for all n.

(2) Let a and b any two real numbers and m a positive integer. If (a-b) is divisible by m, show that $a^n - b^n$ is also divisible by m.

Proof:- For n = 1, it is given that the statement P(n): "$(a^n - b^n)$ is divisible by m." is true. Let P(r) hold.

$\therefore$ $(a^r - b^r)$ is divisible by m.

Now $a^{r+1} - b^{r+1} = (a+b)(a^r - b^r) - ab(a^{r-1} - b^{r-1})$

$= (a+b)(a^r - b^r) - ab(a-b)c$

where c is an integer.

Since R.H.S. of the above identity is divisible by m we conclude that P(r+1) holds.

Hence by induction principle we conclude that $(a^n - b^n)$ is divisible by m for every n.

Applications of Binomial theorem.

(1) Show that $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x-a} = n. a^{n-1}$

Proof - Put $x = a+h$, $h \to 0$

$$\therefore \lim_{x \to a} \frac{x^n - a^n}{x-a} = \lim_{h \to 0} \frac{(a+h)^n - a^n}{a+h-a}$$

$$= \lim_{h \to 0} \frac{(a+h)^n - a^n}{h}$$

But by Binomial theorem

$$(a+h)^n - a^n = \left[a^n + {}^nc_1 a^{n-1} h + - - - + h^n\right] - a^n$$
$$= {}^nc_1\, a^{n-1}.\, h + {}^nc_2\, a^{n-2}\, h^2 + -- + h^n$$

$$\therefore \frac{(a+h)^n - a^n}{h} = {}^nc_1\, a^{n-1} + {}^nc_2\, a^{n-2}\, h + --- + h^{n-1}$$

[∵ h ≠ 0]

$$\therefore \lim_{h \to 0} \frac{(a+h)^n - a^n}{h} = \lim_{h \to 0} {}^nc_1\, a^{n-1} + {}^nc_2\, a^{n-2}\, h + --- + h^{n-1}$$
$$= {}^nc_1\, a^{n-1} = n\, a^{n-1}$$

(2) Prove that $\cos 3\theta = 4 \cos^3\theta - 3 \cos\theta$

and $\sin 3\theta = 3 \sin\theta - 4 \sin^3\theta$

Proof - We know that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$$

This can be proved by using the principle of Mathematical Induction.

$$\therefore (\cos\theta + i \sin\theta)^3 = \cos 3\theta + i \sin 3\theta$$

$$\cos^3\theta + 3i \cos^2\theta \sin\theta + 3i^2 \cos\theta \sin^2\theta + i^3 \sin^3\theta$$
$$= \cos 3\theta + i \sin 3\theta$$

Equating the real and imaginary parts

$\cos^3\theta - 3\cos\theta\sin^2\theta = \cos 3\theta$

i.e. $\cos^3\theta - 3\cos\theta(1-\cos^2\theta) = \cos 3\theta$

i.e. $\cos^3\theta - 3\cos\theta + 3\cos^3\theta = \cos 3\theta$

$\therefore \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Similarly we get

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Some challanging problems

(1) Use mathematical induction to show that

$\frac{d}{dx}(x^n) = nx^{n-1}, n \in N$

Hint:- Show first that $\frac{d}{dx}(x) = 1$

Then use product formula for obtaining the derivative of $x^{r+1} = x^r \cdot x$

(2) Prove by mathematical induction the identity

$(a_1 a_2 \text{---} a_m)^n = a_1^n a_2^n \text{---} a_m^n$

(3) Prove by mathematical induction the identity

$\log(a^n) = n\log a, n \in N$

Where a is any positive real number

Hint: For the 2nd step of induction use the result that

$\log(x-y) = \log x + \log y.$

(4) Find the term independent of x in the expansion of

$\left(\sqrt{\frac{x}{3}} + \frac{3}{2x^2}\right)^{10}$ Ans $\left({}^{10}C_2 \left(\sqrt{\frac{x}{3}}\right)^8 \left(\frac{3}{2x^2}\right)^2 = \frac{5}{4}\right.$

(5) Which number is greater $99^{50} + 100^{50}$ or $(101)^{50}$

Hint :- $101^{50} - 99^{50} = 100^{50}$ + positive terms

Ans. $(101)^{50}$ is greater

(6) Find the value of

$$\frac{18^3+7^3+3\times18\times7(18+7)}{3^6+6\times243\times2+15\times81\times4+20\times27\times8+15\times9\times16+6\times3\times32+64}$$

(Hint:- Use the expansions of $(18+7)^3$ and $(3+2)^6$

Answer = 1)

Chapter Tests

(a) Sample questions for oral test.

(1) What do you mean by "mathematical statement"?

(2) What do you understand by "Induction"?

(3) What is meant by counter example ?

(4) What is the meaning of P(6) if P(n) stands for "n is a prime number". Is P(6) true ?

(Ans. P(6) : 6 is a prime number, P(6) is false)

(5) What do you mean by the term "Binomial"?

)6) Can we expand $(1+x)^{3/7}$? If so under What condition

(Ans. $|x| < 1$)

(7) Can we have two middle terms in a binomial expansions?

Ans. Yes, when index is an odd number

(8) What is the general term in the expansion of $(a+b)^n$?

Why is it called a general term?

Ans. $t_r = {}^nc_{r-1} a^{n-(r-1)} b^{r-1}$)

(9) What is the sum of the coefficients in the expansion of $(a+b)^{10}$ (Ans. 2^{10})

Sample questions for written test:

(1) If P(n) is the statement "n^2-n+17 is prime", Prove that P(1), P(6), P(14) are true, Prove also that P(17) is not true. How does it not contradict the principle of induction?

(2) Give an example of a statement P(n) depending on the integer n such that P(4) is true but P(5) is not true.

(3) Prove by induction the identity.

$$\frac{1}{2} + \frac{1}{2.3} + \frac{1}{3.4} + - - + \frac{1}{n.(n+1)} = \frac{n}{n+1}$$

(4) Draw Pascal's triangle for n=10 and determine the values of ${}^7c_4$, ${}^7c_3$ and ${}^8c_4$ and hence verify that

(i) ${}^7c_3 + {}^7c_4 = {}^8c_4$ (ii) ${}^7c_4 + {}^7c_5$ ~~and~~ $= {}^8c_5$

Generalise this result (Ans. 35, 35, 70. ${}^nc_r + {}^nc_{r+1} = {}^{n+1}c_{r+1}$)

(5) Using $(\cos Q + i\sin Q)^2 = \cos 2Q + i\sin 2Q$

Prove that

(a) $\sin 2Q = 2\sin Q \cos Q$

(b) $\cos 2Q = \cos^2 Q - \sin^2 Q$

(6) If x is a positive real number prove that

$(1+x)^n > (1+ x)$, $n \in N$

(7) Find the term independent of b in the expansion of $(b-\frac{1}{b})^{10}$ (Ans. $-{}^{10}c_5$

(8) Find the approximate value of $(9)^{4/5}$ to 6 decimal places. (Ans. 929268) $-{}^{10}c_5$

(9) Show that when x is small (so that the powers of x more than 4 can be neglected)

$$\sqrt{x^2+4} - \sqrt{x^2+1} = 1 - \frac{x^2}{4} + \frac{7}{64} x^4.$$

CHAPTER - 7

EXPONENTIAL AND LOGARITHMIC SERIES

1. Introduction

7.1 Exponential Series

We have studied about Arithmetic, Geometric and Binomial series. In this chapter we shall study some particular series known as Exponential and Logarithmic series. The following, from earlier chapters, will be made use of:

i) **Geometric Series:**

Let $a_1 + a_2 + a_3$ ------- be a series in which none of a_n's is zero and $\frac{a_{k+1}}{a_k} = r$, a constant (i.e. independent of k) then the series is said to be in G.P. and its sum upto n-terms is given by

$$S_n = \frac{a_1(1-r^n)}{1-r}, \text{ if } r < 1$$

$$= \frac{a_1(r^n-1)}{r-1}, \text{ if } r > 1$$

In case $|r| < 1$ and n is sufficiently large,

$$S_n \to \frac{a}{1-r} \quad \text{as } n \to \infty$$

ii) **Factorial of a non negative integer:**

$\underline{|n} = n(n-1)(n-2)$------------------------3.2.1.

Where n is a non-negative integer Note that $\underline{|0} = 1$.

iii) **Binomial series expansion of $(1+x)^y$:**

$$(1+x)^y = 1 + yx + \frac{y(y-1)}{\underline{|2}}x^2 + \frac{y(y-1)(y-2)}{\underline{|3}}x^3 + \text{-----------}$$

Note that this theorem is valid even when $|x| < 1$ and y a rational number.

iv) <u>The inequality $2^{n-1} \leq \underline{|n}$ for all +ve integers</u>

$2^{n-1} \leq \underline{|n}$ is true for $n = 1$.

Let $2^{n-1} \leq \underline{|n}$ for n

Put n = n+1

$2^{n} \leq \underline{|n+1} \Rightarrow 2.\,2^{n-1} \leq (n+1)\,\underline{|n}$ which is true for $n \geq 1$

since we have assumed $2^{n-1} \leq \underline{|n}$ for n

and $2 \leq \underline{|n+1}$ for n=1.

v) <u>The Complex number, their real and imaginary parts and the formula $e^{iQ} = \text{Cos}Q + i\text{Sin}Q$.</u>

We have already seen that if $a+ib = c+id \Leftrightarrow a=c$ and $b=d$

Similarly,

$e^{iQ} = \text{Cos}Q + i\text{Sin}Q$

The corresponding series for CosQ and SinQ can be obtained by equating real and imaginary parts on both the sides, when e^{iQ} is expressed in the form of series.

vi) <u>The combinatorial Coefficients C(n,r):</u>

$$C(n,r) = \frac{\underline{|n}}{\underline{|r}\;\underline{|n-r}}$$

Also note that $C(n,r) = C(n, n-r)$

vii) The number e is the base of a system of logarithm called the Napierian system (after the name of its inventor John Napier). The base e is mostly used in theoretical investigations.

2. Content Analysis

7.1 Exponential Series:

In this section, the number of each subsection is in, accordance with the Text book.

1) The students may be asked to expand $(1+\frac{1}{n})^n$, using Binomial theorem.

$$(1+\frac{1}{n})^n = 1+n.\ \frac{1}{n} + \frac{n(n-1)}{\underline{|2}}\ .(\frac{1}{n})^2 + \frac{n(n-1)\ (n-2)}{\underline{|3}}\ .(\frac{1}{n})^3 + \text{-------}$$

$$1 + 1 + \frac{1}{\underline{|2}}(1-\frac{1}{n}) + \frac{1}{\underline{|3}}(1-\frac{1}{n})\ (1-\frac{2}{n}) + \text{---------}$$

Now as $n \to \infty$; $\frac{1}{n}, \frac{2}{n}$ etc. all tend to zero

$$\therefore \lim_{n\to\infty} (1+\frac{1}{n})^n = 1+1+\frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \text{--------}$$

The series $1+1+\frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \text{--------}$ is denoted by e

$$\therefore \lim_{n\to\infty} (1+\frac{1}{n})^n = e$$

2. The value of e lies between 2 and 3.

The student may be encouraged to find the value of e. He can conclude that we have

$$e = 1+1+\frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \text{--------}\ \infty$$

$$= 2+\frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \text{----------}\ \infty$$

$= 2+$ (sum of +ve numbers)

$\therefore e > 2$

The student can find that

$\frac{1}{\underline{|3}} < \frac{1}{2^2}$;; $\frac{1}{\underline{|4}} < \frac{1}{2^3}$ etc.

$$e < 1+\ 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \text{--}\ \infty$$

$$< 1+(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\text{------}\ \infty)$$

$< 1 +$ (sum of an infinite G.P. with $a=1$, $r=\frac{1}{2}$)

$< 1 + \frac{1}{1-\frac{1}{2}}$

$< 1 + 2$

< 3 ......(II)

From I & II

$2 < e < 3$.

e lies between 2 and 3

3. <u>The Value of e = 2.7182 (approx.)</u>

As above, student will find the value of some terms upto some places of decimal,

$$e = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots = 2 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots \infty$$

Now $2 = 2.000\ 000$

$\frac{1}{\lfloor 2} = \frac{1}{2} = .500\ 000$

$\frac{1}{\lfloor 3} = \frac{1}{3} \cdot \frac{1}{\lfloor 2} = \frac{1}{3}(.5) = .166\ 667$

and so on

Let the student find some terms and then on adding

$e = 2.7182$ (approx.)

4. <u>Expansion of e^x in the form of a series:</u>

Consider

$$\left[\left(1+\frac{1}{n}\right)^n\right]^x = \left(1+\frac{1}{n}\right)^{nx}$$

Now student may be asked to expand it using Binomial theorem.

$$\left(1+\frac{1}{n}\right)^{nx} = 1+nx.\ \frac{1}{n} + \frac{nx(nx-1)}{\lfloor 2} \cdot \left(\frac{1}{n}\right)^2 + \frac{nx(nx-1)\ (nx-2)}{\lfloor 3} \cdot \left(\frac{1}{n}\right)^3 + \ldots$$

$$= 1+x+x\left(x-\frac{1}{n}\right)\frac{1}{\lfloor 2} + x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right) \cdot \frac{1}{\lfloor 3} + \ldots\ldots\ldots$$

Here the student may be introduced with the concept that as $n \to \infty$; $\frac{1}{n}$, $\frac{2}{n}$ etc. all tend to zero

$$\therefore \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^n\right]^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \dots\dots\dots\dots \infty$$

Here, we may assume $\lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^n\right]^x = \left[\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n\right]^x$

$$\therefore e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \dots\dots\dots\dots\dots\dots, \infty$$

Which is called Exponential series and function e^x is called exponential function.

3. <u>Learning Outcomes</u>

Exponential Series is

$$e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \dots\dots\dots\dots + \frac{x^n}{\lfloor n} + \dots \infty$$

The following results may be developed by the students by replacing x on both the sides;

i) When $x = 0$

$$e^0 = 1 + 0 + 0 + \dots\dots\dots\dots\dots = 1$$

ii) When $x = 1$,

$$e^1 = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \dots\dots\dots$$

iii) $e^{-1} = 1 - \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \dots\dots\dots$

iv) $e + e^{-1} = 2\left(1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 4} + \dots\dots\dots\right)$

v) $e - e^{-1} = 2\left(1 + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 5} + \dots\dots\dots\right)$

vi) When $x = 2$

$$e^2 = 1 + \frac{2}{\lfloor 1} + \frac{2^2}{\lfloor 2} + \frac{2^3}{\lfloor 3} + \dots\dots$$

vii) On adding & subtracting the series for

e^{x} and e^{-x}; we have

$$e^{x} + e^{-x} = 2\left[1+\frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \ldots\ldots\right]$$

$$e^{x} - e^{-x} = 2\left[\frac{x}{\underline{|1}} + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \ldots\ldots\right]$$

4. Teaching strategies

Motivation

i) As we all know that motivation play an important role in teaching learning process. It is needless to say that first teacher must be self motivated. Before introducing a chapter he must have its own planning.

ii) This chapter may be developed through learning-doing activi

iii) As the students have previous knowledge regarding G.P., Binomial theorem and inequality, the lesson can be very well developed by considering expansion of $(1 + \frac{1}{n})^n$ as mentioned in content analysis.

Graph of Exponential function e^x

The students may be motivated to draw graph of e^x.

Let $y = e^x = 1+x+\frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots\ldots\ldots$

Now if $x = 0$, $y = 1$

$\Rightarrow (0,1)$ is a point on the curve. As x increases, each term after the first term increases and tends to ∞ when $x \rightarrow \infty$. $\therefore$ y increases from 1 to ∞ as x increases from 0 to ∞

For negative values of x, y decreases from 1 to 0 as x decreases from 0 to $-\infty$.

Now if we plot the graph of $y=e^x$ for +ve and -ve values of x, we get the curve as shown below:

x	0	1	2	3	-1	-2	-3
e^x	1	2.72	7.39	20.00	0.37	0.14	0.05

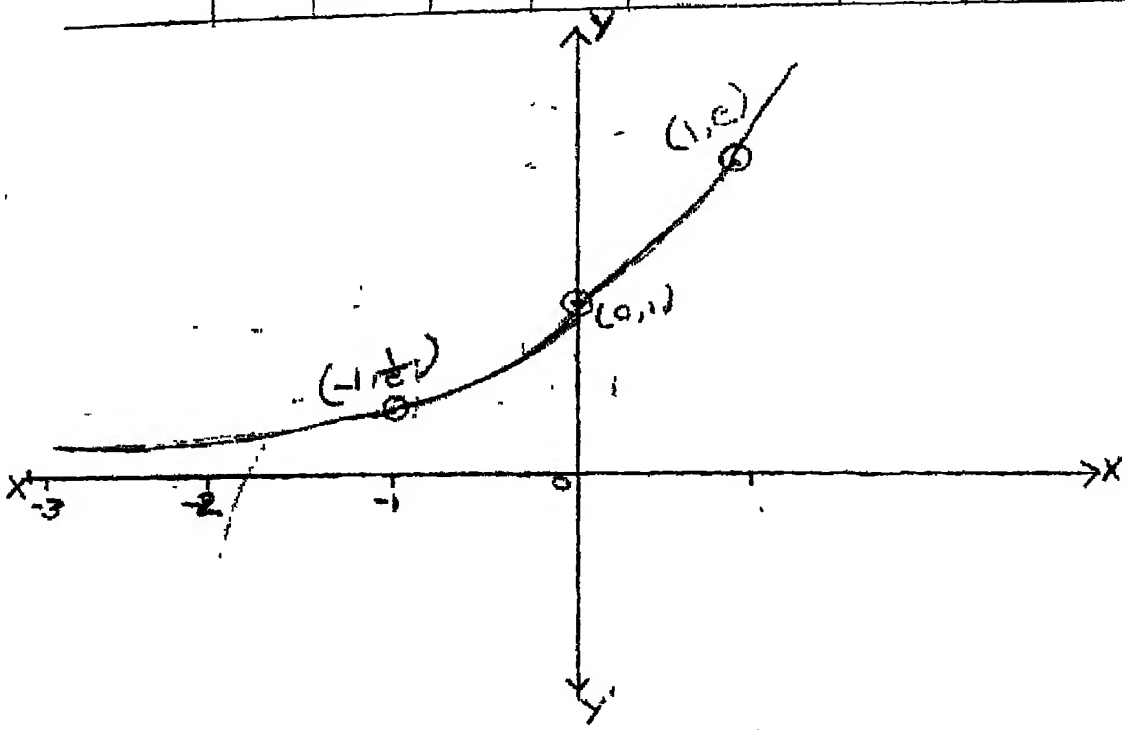


Fig. 7.1(Graph of e^x)

<u>To prove that e is an irrational number:</u>

Let if possible e be equal to a commensable fraction $\frac{p}{q}$ where p and q are positive integers. Then

$$e = 1 + \frac{1}{|\underline{1}} + \frac{1}{|\underline{2}} + \ldots\ldots\ldots + \frac{1}{|\underline{q}} + \frac{1}{|\underline{q+1}} + \ldots\ldots$$

$$\therefore \frac{p}{q} = \left(1 + \frac{1}{|\underline{1}} + \frac{1}{|\underline{2}} + \ldots\ldots + \frac{1}{|\underline{q}}\right) + \frac{1}{|\underline{q+1}} + \ldots\ldots \quad (I)$$

On multiplying both sides of (I) by $|\underline{q}$ then $(q+1)$ terms within the brackets on the R.H.S. of (i) give

$$|\underline{q-1}\cdot p = \text{an integer} + \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \ldots\ldots \quad (II)$$

Now the terms on R.H.S. are +ve and they are also

$$< \frac{1}{q+1} \quad \frac{1}{(q+1)\,(q+2)} + \cdots\cdots$$

$$< \frac{\frac{1}{q+1}}{1 - \frac{1}{q+1}}$$

$$< \frac{1}{q}$$

Which is a proper fraction as q is a +ve integer.

Thus equation (II) shows that $p \,\underline{|q-1}$ is equal to an integer plus a proper fraction which is impossible. SO $e \neq \frac{p}{q}$

Hence e is an irrational number.

Partial Fractions

The operation of decomposing a given proper fraction into simpler fractions is called "Resolution into partial fractions" This is very useful for arriving at certain algebraic results.

For purposes of resolving into partial fractions it will be sufficient to consider only "Proper" fractions i.e. those in which the numerator is of lower degree than the denominator for if this be not the case we can divide out the numerator by the denominator until the remainder is of lower degree than the denominator. Rules are based on the assumption that all the fractions are proper ones. We below give some rules.

Rules

i) To any non repeated linear factor (x-a) in the denominator there corresponds a fraction of the form $\frac{A}{x-a}$ where A is a constant

ii) To a factor $(x-a)^n$ i.e. a linear factor (x-a) repeated n times in the denominator there correspond a grapp of n partial fractions of the form

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \ldots\ldots\ldots\ldots + \frac{A_n}{(x-a)^n}$$

Where A_1, A_2, ...A_n are all constants,

iii) To a non repeated quadratic factor x^2+px+q there corresponds a partial fraction of the form $\frac{Ax+B}{x^2+px+q}$

Let us take an example

Ex. 1.

Resolve $\frac{7x-1}{6x^2-5x+1}$ into partial fractions.

Sol. Here the denominator $= 6x^2-5x+1$

$= (3x-1)\ (2x-1)$

Suppose $\frac{7x-1}{6x^2-5x+1} \equiv \frac{A}{3x-1} + \frac{B}{2x-1}$

Multiplying both sodes by (3x-1) (2x-1) we have

$7x - 1 = A\ (2x-1) + B\ (3x-1)$

It is an identity

$\therefore$ Equating the coefficients of x and the independent terms on the two sides we have

$2A + 3B = 7$ ....(1)

$-A - B = -1$ ....(2)

Solving these we get

$A = -4,\ B = 5$

$$\therefore \quad \frac{7x-1}{6x^2-5x+1} = -\frac{4}{3x-1} + \frac{5}{2x-1}$$

<u>Solutions of problems on Ex. 7.1 of Text book Part-I (Page 135)</u>

Q1. Find the value of e rounded off to one decimal place.

Sol. We have

$$e = 1 + \frac{1}{|\underline{1}} + \frac{1}{|\underline{2}} + \frac{1}{|\underline{3}} + \ldots\ldots\ldots$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \ldots\ldots\ldots$$

$$= 1 + 1 + 0.5 + 0.166 + \ldots\ldots$$

$$\Rightarrow e > 2.707 \qquad (1)$$

Also $e = 1 + \frac{1}{|\underline{1}} + \frac{1}{|\underline{2}} + \frac{1}{|\underline{3}} + \frac{1}{|\underline{4}} + \frac{1}{|\underline{5}} + \frac{1}{|\underline{6}} + \ldots\ldots\ldots$

$$= 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{|\underline{4}}\left[1 + \left(\frac{1}{5} + \frac{1}{5.6} + \frac{1}{5.6.7} + \ldots\ldots\ldots\right)\right]$$

$\because n > 5 \Rightarrow \frac{1}{n} < \frac{1}{5}$; so we have

$$< 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}\left[1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \ldots\ldots\right]$$

$$2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}\left[\frac{1}{1 - \frac{1}{5}}\right]$$

$$2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \cdot \frac{5}{4}$$

$2 + 0.5 + 0.166 + .052 = 2.718$ ....(2)

From (1) and (2), we have

$2.707 < e < 2.718$

$\Rightarrow$ The value of e, rounded off to one decimal place is 2.7

Q2. Find the coefficient of x^n in the expansion of e^{a+bx} in powers of x

Sol. $e^{a+bx} = e^a \cdot e^{bx}$

$$= e^a\left[1 + \frac{bx}{\lfloor 1} + \frac{(bx)^2}{\lfloor 2} + \frac{(bx)^3}{\lfloor 3} + \dots\dots\dots\right]$$

$\therefore$ The term containing $x^n = \dfrac{e^4 (5x)^n}{\lfloor n}$

$\therefore$ Coefficient of $x^n = \dfrac{e^a b^n}{\lfloor n}$

Q3. Find the sum of the $1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 4} + \dots\dots\dots$

Sol. $1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 4} + \dots\dots\dots$

$$= \frac{e + e^{-1}}{2} = \tfrac{1}{2}(e + \tfrac{1}{e})$$

Q4. Find the sum of $\sum_{n=1}^{\infty} \dfrac{n^2}{\lfloor n+1}$

Sol. Here $T_n = \dfrac{n^2}{\lfloor n+1}$

Let $n^2 = A + B(n+1) + C(n+1)n$

on comparing the coefficients of n^2, n and constant we get

$A=1,\ B=-1,\ C=1$

$\therefore n^2 = 1 - (n+1) + (n+1)n$

$$\Longrightarrow \frac{n^2}{\lfloor n+1} = \frac{1}{\lfloor n+1} - \frac{1}{\lfloor n} + \frac{1}{\lfloor n-1}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{\lfloor n+1} = \sum_{n=1}^{\infty}\left[\frac{1}{\lfloor n+1} - \frac{1}{\lfloor n} + \frac{1}{\lfloor n-1}\right]$$

$$\left[\frac{1}{\lfloor\underline{2}} + \frac{1}{\lfloor\underline{3}} + \frac{1}{\lfloor\underline{4}} + \ldots\ldots\ldots\right]$$

$$- \left[\frac{1}{\lfloor\underline{1}} + \frac{1}{\lfloor\underline{2}} + \frac{1}{\lfloor\underline{3}} + \ldots\ldots\ldots\right]$$

$$+ \frac{1}{\lfloor\underline{0}} + \frac{1}{\lfloor\underline{1}} + \frac{1}{\lfloor\underline{2}} + \ldots\ldots\ldots$$

$$(e - 2) - (e - 1) + e$$

$e - 1$ Ans.

Q7. Sum the following series

i) $1 + \frac{2^2}{\lfloor\underline{2}} + \frac{3^2}{\lfloor\underline{3}} + \frac{4^2}{\lfloor\underline{4}} + \ldots\ldots\ldots\ldots\ldots$

Sol. Here $T_n = \frac{n^2}{\lfloor\underline{n}} = \frac{n}{\lfloor\underline{n-1}}$

Putting $n = 1, 2, 3,$

$T_1 = \frac{1}{\lfloor\underline{1-1}}$

$T_2 = \frac{2}{\lfloor\underline{2-1}}$

$T_3 = \frac{3}{\lfloor\underline{3-1}}$

On adding;

$$\sum_{n=1}^{\infty} \frac{n}{\lfloor\underline{n-1}} \quad \frac{1}{\lfloor\underline{1-1}} + \frac{2}{\lfloor\underline{2-1}} + \frac{3}{\lfloor\underline{3-1}} + \frac{4}{\lfloor\underline{4-1}} + \ldots$$

$$1 + \frac{2}{\lfloor\underline{1}} + \frac{3}{\lfloor\underline{2}} + \frac{4}{\lfloor\underline{3}} + \ldots\ldots\ldots$$

$$1 + \frac{(1+1)}{\lfloor\underline{1}} \quad \frac{(1+2)}{\lfloor\underline{2}} \quad \frac{(1+3)}{\lfloor\underline{4}} + \ldots\ldots\ldots$$

$$\left[1 + \frac{1}{\lfloor\underline{1}} + \frac{1}{\lfloor\underline{2}} + \ldots\right] + \left[\frac{1}{\lfloor\underline{1}} + \frac{2}{\lfloor\underline{2}} + \frac{3}{\lfloor\underline{3}} + \ldots\ldots\ldots\ldots\right]$$

$$e + \left(1 + \frac{1}{\lfloor\underline{1}} + \frac{1}{\lfloor\underline{2}} + \ldots\ldots\ldots\right)$$

$e + e = 2e$ Ans.

ii) $1+\dfrac{1+2}{\lfloor 2}+\dfrac{1+2+3}{\lfloor 3}+\dfrac{1+2+3+4}{\lfloor 4}+\ldots\ldots\ldots$

Sol. Here $T_n = \dfrac{1+2+3+\ldots\ldots\ldots+n}{\lfloor n}$

$$= \frac{n(n+1)}{2\lfloor n} = \frac{n+1}{2\lfloor n-1}$$

Now $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2\lfloor n-1} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{n+1}{\lfloor n-1}$

$$= \frac{1}{2}\left[\frac{1+1}{\lfloor 1-1}+\frac{2+1}{\lfloor 2-1}+\frac{3+1}{\lfloor 3-1}+\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$= \frac{1}{2}\left[\frac{2}{\lfloor 0}+\frac{3}{\lfloor 1}+\frac{4}{\lfloor 2}+\ldots\ldots\right]$$

$$= \frac{1}{2}\left[2+\frac{2+1}{\lfloor 1}+\frac{2+2}{\lfloor 2}+\ldots\ldots\ldots\right]$$

$$= \frac{1}{2}\left[\left(2+\frac{2}{\lfloor 1}+\frac{2}{\lfloor 2}+\ldots\ldots\ldots\right)+\left(\frac{1}{\lfloor 1}+\frac{2}{\lfloor 2}+\frac{3}{\lfloor 3}+\ldots\ldots\right)\right]$$

$$= \frac{1}{2}\left[2\left(1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\ldots\ldots\right)+\left(1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\ldots\ldots\ldots\ldots\right)\right]$$

$$= \frac{1}{2}(2e+e) = \frac{3}{2}e \quad \text{Ans.}$$

iii) $\dfrac{2}{\lfloor 3}+\dfrac{4}{\lfloor 5}+\dfrac{6}{\lfloor 7}+\dfrac{8}{\lfloor 9}+\ldots\ldots$

Sol. Here $T_n = \dfrac{2n}{\lfloor 2n+1} = \dfrac{(2n+1)-1}{\lfloor 2n+1} = \dfrac{1}{\lfloor 2n} - \dfrac{1}{\lfloor 2n+1}$

$$\therefore \sum_{n=1}^{\infty}\frac{2n}{\lfloor 2n+1} = \sum_{n=1}^{\infty}\frac{1}{\lfloor 2n} - \sum_{n=1}^{\infty}\frac{1}{\lfloor 2n+1}$$

$$= \left(\frac{1}{\lfloor 2}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 6}+\ldots\ldots\ldots\right)$$

$$-\left(\frac{1}{\lfloor 3}+\frac{1}{\lfloor 5}+\frac{1}{\lfloor 7}+\ldots\ldots\ldots\ldots\right)$$

$$\frac{e+e^{-1}}{2} - 1 - \left(\frac{e-e^{-1}}{2} - 1\right)$$

$$= \frac{1}{e} \quad \text{Ans.}$$

Q8. Sum the series

i) $\sum_{n=2}^{\infty} \frac{C(n,2)}{(n+1)!}$

Sol. Here $T_n = \frac{C(n,2)}{(n+1)!}$ $\qquad \frac{n(n-1)}{2\,(n-1)!}$

Let $n^2-n = A + B\,(n+1) + C(n+1)n$

Comparing the Coefficients of n^2, n, n^0, we have

$$B + C = -1$$
$$C = 1$$
$$A + B = 0$$

Solving these equations

$A=2,\ B=-2,\ C=1$

$$T_n = \tfrac{1}{2}\left[\frac{2 - 2(n+1) + (n+1)n}{(n+1)!}\right]$$

$$= \tfrac{1}{2}\left[\frac{2}{(n+1)!} - \frac{2}{n!} + \frac{1}{(n-1)!}\right]$$

$$\sum_{n=2}^{\infty} \frac{C(n,2)}{(n+1)!} = \tfrac{1}{2}\sum_{n=2}^{\infty}\left[\frac{2}{(n+1)!} - \frac{2}{n!} + \frac{1}{(n-1)!}\right]$$

$$= \tfrac{1}{2}\Big[2\left(\frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots\ldots\right)$$

$$-2\left(\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots\ldots\right)$$

$$+\left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots\right)$$

$$= \tfrac{1}{2}\left[\left(2\left(e-2-\tfrac{1}{2}\right) - 2(e-2) + (e-1)\right)\right]$$

$$= \tfrac{1}{2}\left[(2e-4-1-\ldots 4+e-1)\right]$$

$$= \tfrac{1}{2}\left[(e-2)\right] \text{ Ans.}$$

ii) $$\sum_{n=1}^{\infty} \frac{c(n,o) + C(n,1) + \ldots\ldots\ldots\ldots + C(n,n)}{P\ (n,n)}$$

$$\sum_{n=1}^{\infty} \frac{2^n}{\underline{\lfloor n}}$$

$$= \frac{2}{\underline{\lfloor 1}} + \frac{2^2}{\underline{\lfloor 2}} + \frac{2^3}{\underline{\lfloor 3}} + - - - - - - - - -$$

$$= \left(1 + \frac{2}{\underline{\lfloor 1}} + \frac{2^2}{\underline{\lfloor 2}} + \frac{2^3}{\underline{\lfloor 3}} + \ldots\ldots\ldots\ldots\right) - 1$$

$e^2 - 1$ Ans.

Q 10 Sum the series: $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{n!}$

Sol. $$\sum_{n=1}^{\infty} \frac{2n}{\underline{\lfloor n}} = \sum_{n=1}^{\infty} \frac{2}{\underline{\lfloor n-1}}$$

$$= 2 \sum_{n=1}^{\infty} \frac{1}{\underline{\lfloor n-1}}$$

$$= 2\left(1 + \frac{1}{\underline{\lfloor 1}} + \frac{1}{\underline{\lfloor 2}} + \ldots\ldots\ldots\ldots\right)$$

$$= 2e \quad \text{Ans.}$$

Additional Exercise

Prove that

1. $e^{-1} = \frac{2}{\underline{\lfloor 3}} + \frac{4}{\underline{\lfloor 5}} + \frac{6}{\underline{\lfloor 7}} + \ldots\ldots\ldots\ldots$

2. $1 + \frac{4^2}{\underline{\lfloor 3}} + \frac{4^4}{\underline{\lfloor 5}} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{1}{8}\left(e^4 - e^{-4}\right)$

3. $1 + \frac{1+3}{\underline{\lfloor 2}} + \frac{1+3+3^2}{\underline{\lfloor 3}} + \frac{1+3+3^2+3^3}{\underline{\lfloor 4}} + \ldots\ldots = \frac{1}{2}e\left(e^2 - 1\right)$

4. Find the value of
$a^2 - b^2 + \frac{1}{\underline{\lfloor 2}}\left(a^4 - b^4\right) + \frac{1}{\underline{\lfloor 3}}\left(a^6 - b^6\right) + \ldots\ldots\ldots\ldots \infty$

5. Expand

i) $\dfrac{e^{5x}+e^{x}}{e^{3x}}$ in a series of ascending powers of x.

$$\left(\text{Ans. } 2\left(1+\frac{2^2x^2}{\lfloor 2}+\frac{2^4x^4}{\lfloor 4}+\ldots\ldots\ldots\right)\right)$$

ii) $\frac{1}{2}(e^{ix}-e^{-ix})$ in a series of ascending powers of x.

$$\left(\text{Ans. } 1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}-\ldots\ldots\ldots\ldots\infty\right)$$

6. Show that

$$1+\frac{1+2}{\lfloor 2}+\frac{1+2+2^2}{\lfloor 3}+\frac{1+2+2^2+2^3}{\lfloor 4}+\ldots\ldots=e^2-e$$

7. $$1+\frac{2^3}{\lfloor 2}+\frac{3^3}{\lfloor 3}+\frac{4^3}{\lfloor 4}+\ldots\ldots\ldots\ldots\ldots\ldots=5e$$

8. Find the coefficient of x^n in $(1-3x+x^2)/e^x$

9. Prove that

$$\frac{1^2.2^2}{\lfloor 1}+\frac{2^2.3^2}{\lfloor 2}+\frac{3^2.4^2}{\lfloor 3}+\ldots\ldots\ldots\ldots=27e$$

10. $$\sum_{n=1}^{\infty}\frac{1^2+2^2+3^2+\ldots+n^2}{\lfloor n}=\frac{17}{6}e$$

7.2 Logarithmic Series:

Introduction

The students have studied in earlier classes that if a, x, N be three quantities connected by the relation $a^x = N$, then a is called the base and x is called the logarithm of N to the base a and is denoted by $\log_a N$. Thus the logarithm of a number to a given base is the index of the power to which the base must be raised to obtain the number, that is, the two equations given below represent the same relation

$$a^x = N \qquad (1)$$

$$x = \mathrm{Log}_a N \qquad (2)$$

The relation (2) is read as x = logarithm of N to base a.

From (1) and (2) we derive another identity which is very useful and should be remembered

$$N = a^{\log_a N}$$

2. Content Analysis :

7.2 In this section, the number of each subsection is in accordance with the text book.

Also, in this section we will take the number e as the base of a logarithm, whenever the base is not explicitly mentioned. Here we obtain an expression for $\log(1+x)$ as a series of powers of x. This expression is valid only when $|x| < 1$

Expansion of a^x:

Writing e for x,

$$e^{cx} = 1 + cx + \frac{c^2x^2}{\underline{|2}} + \frac{c^3x^3}{\underline{|3}} + \dots\dots\dots\dots \quad (I)$$

Let $e^c = a \Rightarrow c = \log_e a$

substituting for c in (I) we, have;

$$a^x = 1 + x \log a + \frac{x^2}{\underline{|2}}(\log a)^2 + \frac{x^3}{\underline{|3}}(\log a)^3 + \dots\dots$$

This expression of a^x in terms of x is called Exponential theorem.

ii) Using, Exponential and Binomial theorem a student can obtain logarithmic series as follows:

If $a > 0$, by Exponential theorem

$$a^y = 1 + y \log a + \frac{y^2}{\underline{|2}}(\log a)^2$$

Putting $a = 1+x$, we get

$$(1+x)^y = 1 + y \log(1+x) + \frac{y^2}{\underline{|2}}[\log(1+x)]^2 + \dots\dots\dots\dots\dots\dots (II)$$

But if $|x| < 1$, By Binomial Theorem

$$(1+x)^y = 1 + yx + \frac{y(y-1)}{\underline{|2}}x^2 + \frac{y(y-1)(y-2)}{\underline{|3}}x^3 + \dots\dots\dots\dots (III)$$

From (II) and (III), For $|x| < 1$, we have

$$1 + y\log(1+x) + \frac{y^2}{\underline{|2}}\left[\log(1+x)\right]^2 + \dots\dots = 1 + yx + \frac{y(y-1)}{\underline{|2}}x^2 + \dots\dots$$

It is an identity and so equating the coeficients of y on both sides, we get

$$\operatorname{Log}(1+x) = x + \frac{(-1)}{\underline{|2}}x^2 \cdot\cdot \frac{(-1)(-2)}{\underline{|3}}x^3 + \frac{(-1)(-2)(-3)}{\underline{|4}}x^4 + \dots\dots\dots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots\dots$$

This is called Logarithmic Series.

3. <u>Learning Outcomes:</u>

i) Logarithmic Series is

$$\text{Log}(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots\dots\dots \infty$$

The following results may be developed by the students by replacing x on both the sides of (i), we get.

ii) If $|x|$ is not loss than 1, the series $\text{Log}(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \infty$ is not valid.

iii) If $x = 1$, then

$\text{Log } 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\dots\dots$ is valid

iv) If $x = 1$, then the series has no sum.

v) If $x=2$, then the series is

$2 - \frac{2^2}{2} + \frac{2^3}{3} - \dots\dots$ and it cann't have a sum

vi) Changing x into $-x$; we get

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots\dots\dots$$

vii) On subtracting (i) & (vi);

$$\log(1+x) - \log(1-x) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\dots\dots\right]$$

$$\Rightarrow \log\frac{1+x}{1-x} = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\dots\dots\right]$$

viii) In the result of (vi), put $\frac{1+x}{1-x} = \frac{n+1}{n}$

$$\Rightarrow x = \frac{1}{2n+1}$$

$$\therefore \log(n+1) - \log n = 2\left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \cdots\right]$$

This series is useful in the construction of logarithmic tables.

3. **Teaching Strategies:**

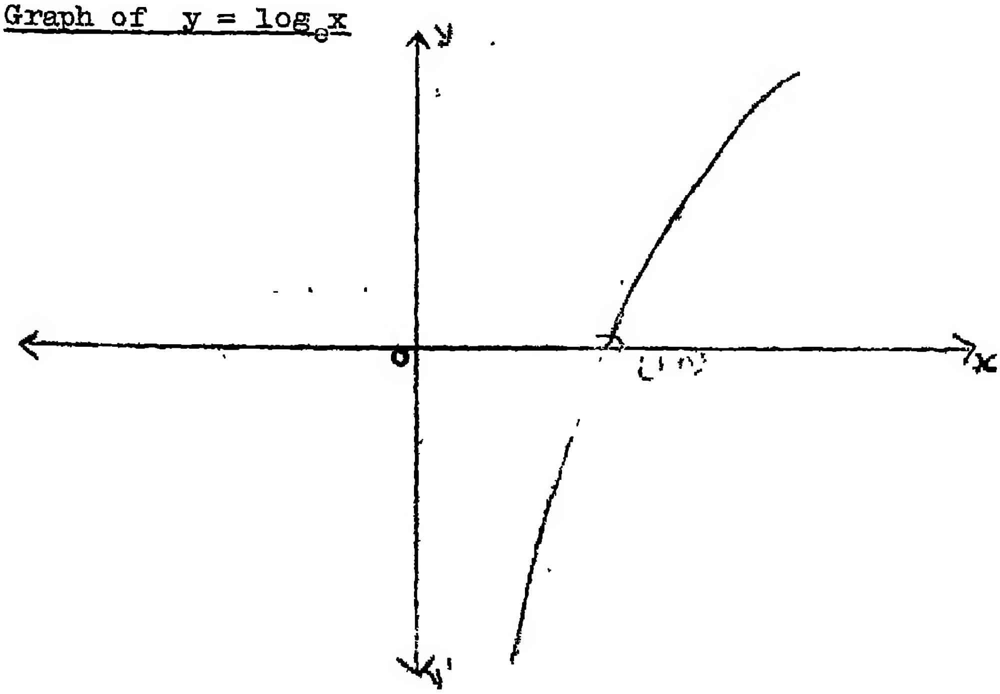
Motivation

In a class room, there are two parties, one who wants to give something while the other who receives the things If both parties are ready, then certainly the teaching will be effective. So it requires motivation from both sides. A Teacher has an important part in presenting a lesson by involving the students much.

As the students have a knowledge of exponential and Binomial theorem then logarithmic series can be developed by following steps as shown in content analysis.

Additional

i) Graph of $y = \log_e x$



Graph between e^x and $\log_e x$

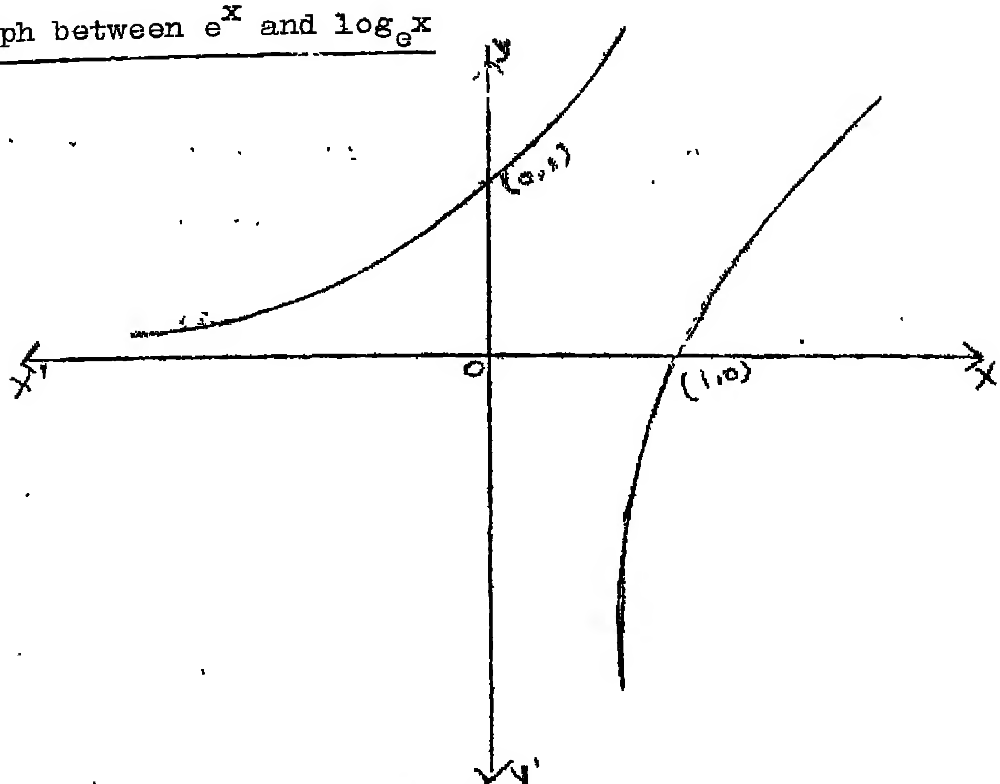


Fig. 7.3

ii) Calculation of Napierian Logarithms

Logarithms of numbers to the base e are called Napierian or Natural logarithms. Now from series viii) of last remark, we have by putting n=1

$$\log_e 2 = 2\left[\frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5.3^5} + \ldots\ldots\ldots\right]$$

$$= 2\left[.333333 + \tfrac{1}{3} \times .037037 + \frac{1}{5} \times .004115 + \ldots\ldots\ldots\right]$$

$$= 2 \times .346573$$

$$= .693146$$

Again putting n=2 in the same series, we get

$$\log_e 3 = \log_e 2 + 2\left[\frac{1}{5} + \frac{1}{3.5^3} + \frac{1}{5.5^5} + \ldots\ldots\right]$$

$$= .693146 + .405465$$

$$= 1.09861$$

Proceeding like we can find the Napierian logarithms of any number. Thus we shall find that Log 10 = 2.30258509.

Conversion from Napierian logarithms into Common Logarithms:

Logarithms to base 10 are known as common logarithms.

We know that

$$\mathrm{Log}_{10}\, n = \mathrm{Log}_e n \times \frac{1}{\mathrm{Log}_e 10}$$

Thus the common logarithms of any number to the base 10 is obtained by multiplying its Napierian logarithms by the constant factor $\frac{1}{\mathrm{Log}_e 10}$ which is called the modulus of the common system. This modulus is denoted by greek letter μ and its value

$$= \frac{1}{\mathrm{Log}_e 10} = \frac{1}{2.3025850} = .43429448$$

Thus we have from series viii) above $\mathrm{Log}_{10}(n+1) - \log_{10} n = \mu \, \mathrm{Log}_e(n+1) - \mu \, \mathrm{Log}_e n$

$$= 2\mu \left\{ \frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots\dots\dots \right\}$$

With the help of this formula, we can obtain the common logarithms of any number:

Note 1: These formulae are required for the calculation of the logarithms of prime numbers only for the logarithms of a composite number can be obtained by adding together the logarithms of its prime factors.

Note 2: All logarithms were first found to the base e and then converted to the base 10 with the help of the conversion formula. That is why Napierian logarithms are called Natural logarithms.

Solution of problems on Ex. 7.2 page 142.

Q.1 Prove that $\log 2 < 1 < \log 3$

Sol. We know that $e \approx 2.7$

$\therefore 2 < 2.7 < 3$

$2 < e < 3$

Taking log

$\text{Log } 2 < \log e < \log 3$ [$\because$ log x increases as x increases & $\log e = 1$.

$\Rightarrow \log 2 < 1 < \log 3$

<u>Aliter</u>: $e \approx 2.7 \Rightarrow e \approx \frac{27}{10} \Rightarrow 10e \approx 27 \Rightarrow \frac{10e}{27} \approx 1$

$\Rightarrow \frac{20e}{27} \approx 2$

and $\frac{30e}{27} \approx 3$

$$\text{Log}2 = \log \frac{20e}{27} = \log e + \log \frac{20}{27}$$
$$= 1 + \log\left(1 - \frac{7}{27}\right)$$
$$= 1 - \left[\frac{7}{27} + \frac{7^2}{2\times 27^2} + \ldots\ldots\ldots\ldots\right]$$
$$< 1 \quad [\because \text{ Terms in brackets are all +ve})$$

Also $$\log 3 = \log \frac{30e}{27} = \log\left(\frac{10e}{9}\right)$$
$$= \log e + \log \frac{10}{9} = 1 + \log\left(1 + \frac{1}{9}\right)$$
$$= 1 + \left(\frac{1}{9} - \frac{1}{2\times 9^2}\right) + \frac{1}{3 \times 9^3} + \frac{1}{4 \times 9^4} + \ldots\ldots)$$
$$= 1 + \left[\frac{1}{9}\left(1 - \frac{1}{18}\right) + \frac{1}{9^3}\left(\frac{1}{3} - \frac{1}{36}\right) + \ldots\ldots\right]$$

$\therefore \text{Log}2 < 1 < \text{Log } 3$ [$\because$ The terms in brackets are positive)

Q2. If $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$ and if $|x|<1$

Prove that $x = y + \frac{y^2}{\lfloor 2} + \frac{y^3}{\lfloor 3} + \ldots\ldots$

Sol. $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$

$= \log_e (1+x)$

$\therefore$ By definition of logarithm $\quad 1 + x = e^y$

$\Rightarrow 1 + x = 1+y + \frac{y^2}{\lfloor 2} + \frac{y^3}{\lfloor 3} +$

$\Rightarrow x = y + \frac{y^2}{\lfloor 2} + \frac{y^3}{\lfloor 3} + \ldots\ldots$

Q.E.D.

Q3. Prove that the series $\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \ldots\ldots$ has the same sum as the series $\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\ldots$

Sol. We know that

$\log (1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\ldots$

$\Rightarrow -\log (1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots$

Putting $x = \frac{1}{n+1}$

$\therefore -\log (1 - \frac{1}{n+1}) = \frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \ldots\ldots$

or $\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + - - - - = -\log (\frac{n}{n+1})$

$= -\log n + \log(n+1)$

$= \log \frac{n+1}{n}$

$= \log (1 + \frac{1}{n})$

$= \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\ldots$

Q.E.D.

Q4. Prove that

$$\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\ldots\ldots=\log 2-\tfrac{1}{2}$$

Sol. $T_n=\frac{1}{(2n-1).2n(2n+1)}\equiv\frac{A}{2n-1}+\frac{B}{2n}+\frac{C}{2n+1}$

$$\equiv\frac{A(2n+1).2n+B(2n-1)(2n+1)+C2n(2n-1)}{(2n-1).2n.(2n+1)}$$

$\therefore$ Comparing Coefficients of n^2, n, constant in the numerator.

We get

$4A+4B+4C=0 \Longrightarrow A+B+C=0$

$2A-2C \quad = \quad 0 \Longrightarrow A-C=0$

$-B=1 \Longrightarrow B=-1$

Solving this system of equations we get

$A=\tfrac{1}{2},\ B=-1,\ C=\tfrac{1}{2}$

$\therefore$ $T_n=\frac{1}{2(2n-1)}-\frac{1}{2n}+\frac{1}{2(2n+1)}$

$=\tfrac{1}{2}\left[\frac{1}{2n-1}-\frac{1}{n}+\frac{1}{2n+1)}\right]$

$\therefore$ $T_1=\tfrac{1}{2}\left[1-\frac{1}{1}+\tfrac{1}{3}\right]$

$T_2=\tfrac{1}{2}\left[\tfrac{1}{3}-\tfrac{1}{2}+\frac{1}{5}\right]$

$T_3=\tfrac{1}{2}\left[\frac{1}{5}-\tfrac{1}{3}+\frac{1}{7}\right]$

$\therefore$ $\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\ldots=T_1+T_2+T_3+\ldots\ldots\ldots$

$$=\frac{1}{2}\left[\left(1-\frac{1}{1}+\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{3}+\frac{1}{7}\right)+\ldots\ldots\ldots\right]$$

$$=\frac{1}{2}\left[\left(1-\frac{1}{2}-\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}-\frac{1}{4}+\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}-\frac{1}{6}+\frac{1}{7}\right)+\ldots\ldots\right]$$

$$=\frac{1}{2}\left[\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots\ldots\right)-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\ldots\right]$$

$$=\frac{1}{2}\left[\log 2+\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\right)-1\right]$$

$$=\frac{1}{2}\left[\log 2+\log 2-1\right]$$

$$=\log 2-1/2$$

Q.E.D

Q5. Prove that

$$\log(1+x)^{1+x}(1-x)^{1-x}=2\left[\frac{x^2}{1.2}+\frac{x^4}{3.4}+\frac{x^6}{5.6}+\ldots\ldots\right]$$

Sol. L.H.S. $\log\left[(1+x)^{1+x}(1-x)^{1-x}\right]=\log(1+x)^{1+x}+\log(1-x)^{1-x}$

$$=(1+x)\log(1+x)+(1-x)\log(1-x)$$

$$=(1+x)\left[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots\ldots\right]$$

$$=-(1-x)\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\right]$$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\text{------}$$

$$+\ x^2-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{4}+\frac{x^6}{5}\text{------}$$

$$-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\ldots\ldots$$

$$+x^2+\frac{x^3}{2}+\frac{x^4}{3}+\frac{x^5}{4}+\ldots\ldots\ldots$$

on cancelling:

$$=2\left[\left(x^2-\frac{x^2}{2}\right)+\left(\frac{x^4}{3}-\frac{x^4}{4}\right)+\left(\frac{x^6}{5}-\frac{x^6}{6}\right)+\ldots\ldots\right]$$

$$=2\left[\frac{x^2}{1.2}+\frac{x^4}{3.4}+\frac{x^6}{5.6}+\ldots\ldots\ldots\right]=\text{R.H.S.}$$

Q.E.D.

Q6. Prove that $2 \log x - \log (x+1) - \log (x-1)$

$$= \frac{1}{x^2} + \frac{1}{2x^4} + \frac{1}{3x^6} + \ldots$$

Sol. L.H.S. $= 2 \log x - [\log (x+1) + \log (x-1)]$

$$= \log x^2 - \log [(x+1)(x-1)]$$

$$\log x^2 - \log (x^2-1)$$

$$\log \frac{x^2}{x^2-1} = -\log \left(\frac{x^2-1}{x^2}\right) = -\log \left(1-\frac{1}{x^2}\right)$$

$$= \frac{1}{x^2} + \frac{1}{2x^4} + \frac{1}{3x^6} + \ldots\ldots$$

= R.H.S.

Q.E.D.

Q7. Find the value of log 4 correct to one decimal place.

Sol. We know that

$$e \approx 2.7$$

$$\Rightarrow e \approx \frac{27}{10} \Rightarrow 10e \approx 27$$

$$\Rightarrow \frac{10e}{27} \approx 1$$

$$\Rightarrow \frac{40e}{27} \approx 4$$

$$\log 4 \approx \log \left(\frac{40e}{27}\right)$$

$$= \log e + \log \frac{40}{27}$$

$$= \quad 1 + \log\left(1 + \frac{13}{27}\right)$$

$$= \quad 1 + \frac{13}{27} - \frac{13^2}{2.27^2} + \ldots\ldots$$

$$\approx \quad 1 + 0.48 - 0.12 = 1.36, \text{ or } \log 4 = 1.4$$

Difference between exponential and logarithmic Series

There are three major differences between the exponential series and the logarithmic series.

1. In the series $e^x = 1 + \frac{x}{|1} + \frac{x^2}{|2} + \frac{x^3}{|3}$ all terms carry +ve signs. In the series $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \ldots$, the terms carry alternately +ve and -ve signs.

2. In the logarithmic series, the factorial symbol does not occur. But in exponential series, the denominators of the terms involve the factorials.

3. The exponintial series is valid for all values of x. The logarithmic series is valid when $|x| < 1$

ADDITIONAL EXERCISE

Prove that

i) $\log_e 3 = 1 + \frac{1}{3.2^2} + \frac{1}{5.2^4} + \frac{1}{7.2^6} + \ldots\ldots$

ii) $\log_e 10 = 3\log_e 2 + \frac{1}{4} - \frac{1}{2}\left(\frac{1}{4}\right)^2 + \frac{1}{3}\left(\frac{1}{5}\right)^3 - \ldots$

iii) $\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \frac{4}{5}x^5 + \ldots = \frac{x}{1-x} + \log(1-x)$

iv) $\frac{x-1}{x+1} + \frac{x^2-1}{2(x+1)^2} + \frac{x^3-1}{3(x+1)^3} + \ldots\ldots = \log x$

v) $1 + \frac{\log_e 2}{|2} + \frac{(\log_e 2)^2}{|3} + \ldots\ldots = \frac{1}{\log_e 2}$

vi) Log $(1+3x+2x^2) = \log(1+x)(1+2x)$

vii) $1 - \log_e 2 = \frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \ldots\ldots\ldots$

viii) Show that the coefficient of x^n in the expansion of

$\log \dfrac{1}{1-x-x^2+x^3}$ in a series of ascending powers

of x is $\frac{1}{n}$ or $\frac{3}{n}$ according as n is odd or even.

ix) Show that

$$\left(\frac{a-b}{a}\right) + \frac{1}{2}\left(\frac{a-b}{a}\right)^2 + \frac{1}{3}\left(\frac{a-b}{a}\right)^3 + \ldots\ldots\ldots$$

$$= \log_e a - \log_e b$$

x) Prove that

$$\frac{1}{2}\log_e 2 = \frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5.3^5} + \frac{1}{7.3^7} + \ldots\ldots\ldots$$

xi) If α and β are the roots of the equation $x^2 - px + q = 0$

then show that $\log(1+px+qx^2) = (\alpha+\beta)x - \frac{(\alpha^2+\beta^2)}{2}$

$$+ \frac{\alpha^3+\beta^3}{3}x^3 - \ldots\ldots\ldots$$

xii) Prove that if

$$f = \frac{x}{1+x^2} + \frac{1}{3}\left(\frac{x}{1+x^2}\right)^3 + \frac{1}{5}\left(\frac{x}{1+x^2}\right)^5 + \ldots\ldots\ldots$$

and $g = x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 - \frac{2}{9}x^9 + \ldots\ldots\ldots$

then $f = g$

5. CHAPTER TEST:

ORAL TEST:

1. Define the number e.

2. What is the approx. value of e upto 4 places of decimal.

3. State the expansion of

i) e, (ii) e^x; (iii) e^0, (iv) e^1, (v) e^{-1}

vi) e^{-x}, vii) $\frac{e^x + e^{-x}}{2}$, viii) $\frac{e^x - e^{-x}}{2}$, ix) $\frac{e^x+e^{-1}}{2}$,

x) $\frac{e-e^{-1}}{2}$

xi) log (1+x) xii) log (1 - x)

xiii) $\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$ xiv) log 2

4. When the series for log (1+x) is not valid.

5. What are the major differences between exponential and logarithmic series.

Written Test

Questions

Q1. Prove that

$$e^x = 1 + x + \frac{x^2}{\lfloor\underline{2}} + \frac{x^3}{\lfloor\underline{3}} + \ldots\ldots\ldots \infty$$

2. Find the sum of infinite series

$$1 + \frac{2^3}{\lfloor\underline{2}} + \frac{3^3}{\lfloor\underline{3}} + \frac{4^3}{\lfloor\underline{4}} + \ldots\ldots\ldots$$

3. Find the value of e upto 5 places of decimal

4. Show that

$$e = \frac{2}{\lfloor\underline{1}} + \frac{4}{\lfloor\underline{3}} + \frac{6}{\lfloor\underline{5}} + \ldots\ldots\ldots \infty$$

5. Sum the series from n = 1 to ∞, when n^{th} term is $\frac{1}{\lfloor\underline{2n+1}}$

6. Obtain logarithmic Series

7. Prove that

$$\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)^2 = \frac{2x}{1+x^2} + \frac{1}{3}\left(\frac{2x}{1+x^2}\right)^3 + \frac{1}{5}\left(\frac{2x}{1+x^2}\right)^5$$

8. $$2\log_e x - \log_e(x+1) - \log_e(x-1) = \frac{1}{x^2} + \frac{1}{2x^4} + \frac{1}{3}x^6$$

9. $$\log\frac{4}{e} = \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \frac{1}{4.5} + \ldots\ldots$$

10. $$\log 2 + 2\left(\frac{1}{5} + \frac{1}{3}\cdot\frac{1}{5^2} + \frac{1}{5}\cdot\frac{1}{5^3} + \ldots\ldots \infty\right) = \log 3$$

Reference books

1. Higher Algebra -by Bornard and Child

2. Higher Algebra - by H.S. Hall & S.R. Knight

3. Algebra - by Manickavachagom Pillai Natrajan.

CHAPTER - 8

CARTESIAN SYSTEM OF RECTANGULAR COORDINATES

1. Introduction

Students have had earlier exposure to elements of synthetic geometry through deduction of geometrical ideas from assumptions and theorems based on <u>undefined</u> and defined terms & rules of logic, to algebraic manipulations and to drawing of graphs in 1-space and 2-space.

Now an attempt is to be made to introduce the students to the use of algebraic methods in establishing geometrical ideas so as to help them to appreciate the beauty and importance of the analytic method in geometry, as a preparation for calculus, as it happened in the history of mathematics.

Analytic (or analytical) geometry is not a branch of geometry but a method of solving geometrical problems with the help of algebra. A branch of analytic geometry is cartesian coordinate geometry with which we are concerned here. There was a time when algebraic processes were treated as an end in themselves without employing them as tools of fruitful investigation in geometry, as could be seen even today by putting questions of the type "Find the equation of ...." more frequently than those of the type "Prove the property of the curve....."

In analytic geometry, one gets a choice to use a synthetic, analytic or a mixed proof in geometric investigation, whichever is easier or more suitable, depending on one's judgement.

Regarding the historical background of analytic geometry, students may be asked to read the pages 392 to 394 of their text book.

In the analytic approach to geometry, it is important to note that a point on a line is defined to be a real number and hence as an element of R and a point of the plane as an element of R X R or R^2, Also, a line in R^2 is a cartesian type of subset and so can be given as

$$\{(x,y) \in R^2 \ / \ ax+by+c=0\}$$

Geometric intuition is availed off in formulating the definiteness in analytic geometry. Chapter-I of this book deals with sets, relations & functions and the chapters 8 to 12 can and may better be treated as its continuation.

Co-ordinate geometry emerges as the study of properties of loci such as lines and curves by means of algebraic treatment of corresponding relations.

The relation or locus concept is a comprehensive one. Associating locus with the path of a moving point under some condition(s) is so narrow that it is no more considered so; but only as any subset of real number pairs (x,y) or of points in a plane, specified by a single equation or inequation. For instance, $x^2+y^2=1$ is the locus of points on a circle, whereas $x^2+y^2<1$, the locus of points inside the circle and $x^2+y^2>1$, the locus of points outside the circle.

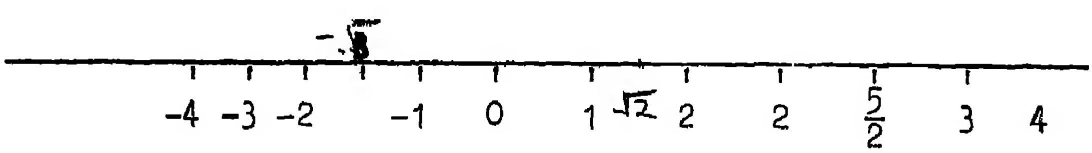
A beginning is made with the introduction of co-ordinates, particularly rectangular, distance between two points, division of a line segment in a given ratio, inclination and slope of a straight line and the angle that one line makes with the other of these, the distance and slope concepts are most frequently used and should form part of the minimum competence expected from a student.

2. Content Analysis

In this section the number of each sub-section is in accordance with that in the text-book.

8.2 Cartesian Coordinate System - The Number Plane

We are anquainted with the fact that each point on the number line has one-one correspondence with every member of R.



Likewise every point of a plane has one-one correspondence with every member of R X R. Thus taking x-axis and y-axis mutually intersecting at right angles at the point O, the origin, we determine the co-ordinates of each point of the plane.

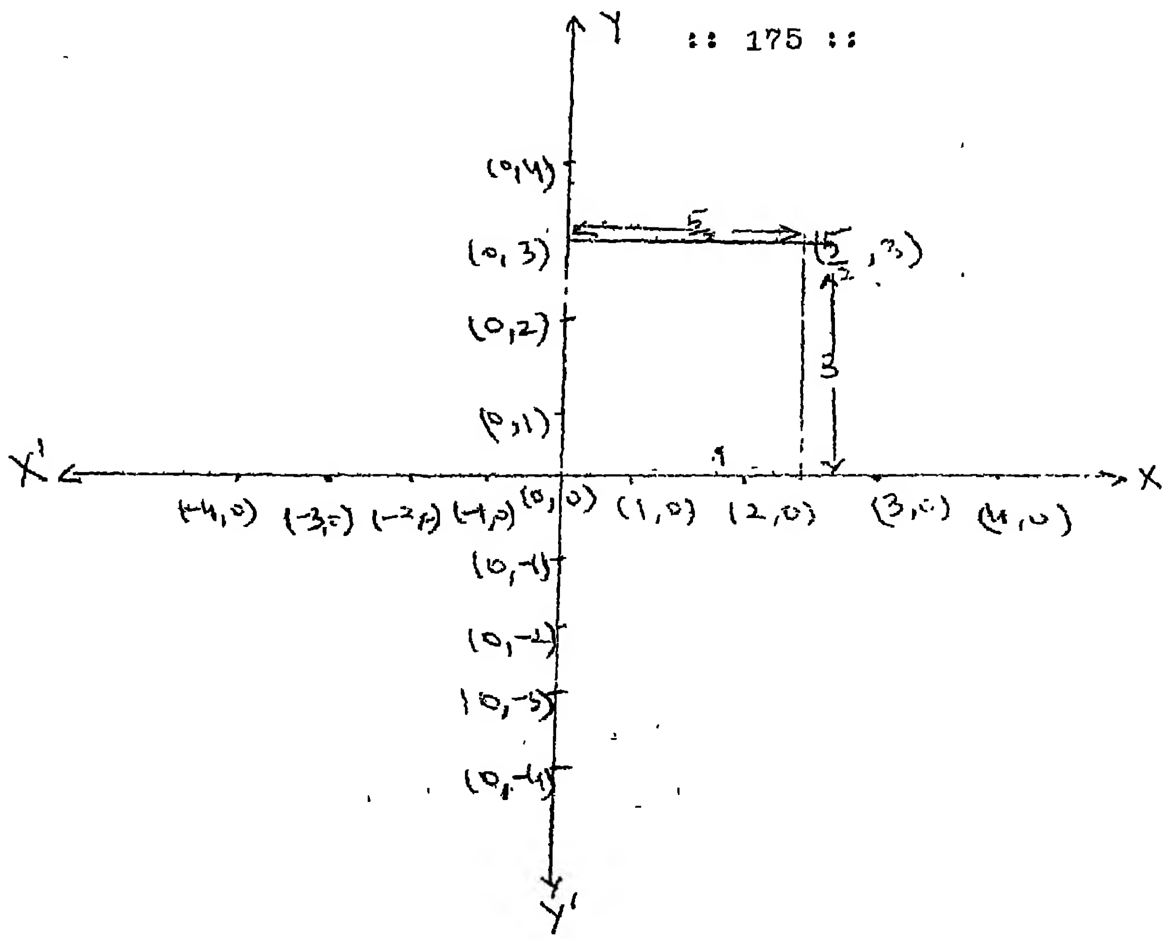


Fig. 8.1

Note that $(x,y) \neq (y,x)$ unless $x=y$

8.3 <u>Distance Formula</u>:

It should be made clear to the students that we are not considering the directed distance between two points. The distance between points (x_1,y_1) and (x_2,y_2) is given by

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

Note: Traditionally the first quadrant is chosen to prove it. This creates a misconception and so needs to be avoided. To pinpoint it, the proof is given below:

Proof: Case 1: PQ not parallel to either axis

Draw lines through P and Q parallel to the X-axis and the y-axis respectively. Let the point of intersection be R

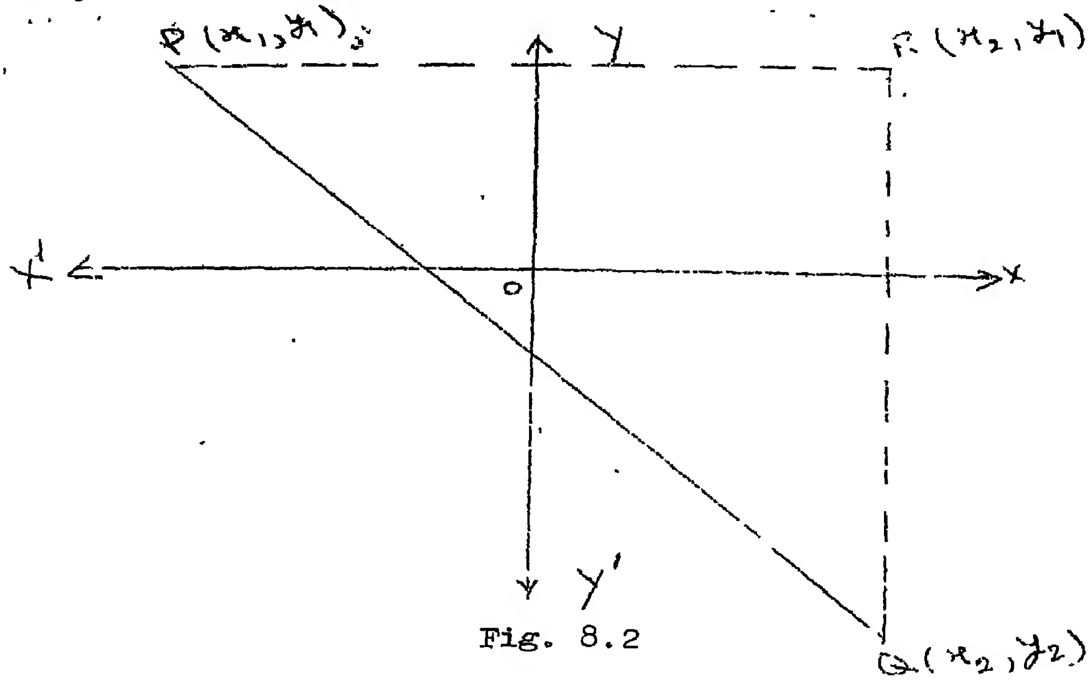


Fig. 8.2

and it is $(x_2 y_1)$. Using pythagoras theorem.

$$PQ^2 = PR^2 + QR^2$$

$$PQ^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

$$\therefore \quad PQ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

Case-II: PQ Parallel to the x-axis

Then $y_2 = y_1$ or $y_2 - y_1 = 0$

$$\therefore \quad PQ = \sqrt{(x_2-x_1)^2 + (0)^2} = |x_2 - x_1|$$

Case-III: PQ parallel to the y-axis

Then $x_2 = x_1$ or $x_2 - x_1 = 0$

$$\therefore \quad PQ = \sqrt{(0)^2 + (y_2 - y_1)^2} = |y_2 - y_1|$$

3. Learning Outcomes

a. Essential Learning Outcomes For All

i) Ability in applying distance formula.

ii) Competence to find the point which divides a given line segment internally or externally in a given ratio.

iii) Competence to find the midpoint of the line joining two given points.

iv) Ability to find the area of a triangle whose vertices are given.

v) Ability to verify the collinearity of three given points.

vi) Ability to find out slope of a straight line and hence to find and verify the conditions of parallelism and perpendicularity of two straight lines.

vii) Ability to find the equation of the locus of a point having a specific property.

b. Learning outcomes for the higher group

i) Given two points $P(x_1,y_1)$ and (x_2,y_2), to find the coordinates of any point lying on/outside (produced on either side) PQ.

ii) To show that if a, b, c, d, $k \in R$, then $\{(x,y)/x=a+bk,\ y+c+dk\}$ is a straight line.

4. **Teaching Strategies:**

Motivation

In order to arouse the interest of the students, motivational activities of the following sort are suggested:

i) One vertex of a square is at (4,-4) and the diagonals intersect at the origin. Draw such a square and

a. Write the coordinates of the other three vertices

b. What is the length of each side of the square ?

c. What is the length of each diagonal of the square?

ii) The midpoint of AB is the origin and the coordinates of A are (-4,2). What are the coordinates of B?.

iii) What is the slope of the line which passes through (2,7) and

a. is parallel to X-axis

b. passes through origin

c. is parallel to the y-axis

iv) Given two points A(2,5) and B(4,-1). Find some more pairs of points which are at the same distance as AB.

v) Given two points A(2,5) and B(4,-1). Keeping A fixed, find some more points each of which is at the same distance as AB.

vi) Can you suggest what is the locus of B in activity (v)?.

Misconceptions and Common Errors

i) Impress upon the students that the slope of lines parallel to the Y-axis is not defined. In other words $\frac{y_2-y_1}{x_2-x_1}$ holds good only when $x_2 \neq x_1$.

ii) It is essential to note the difference between dividing a line segment in the ratio 2:3 and in locating a point $\frac{2}{3}$ distance from one end of the segment.

Solutions/Hints for difficult problems

Exercise 8.1

Q5. Hint: Let the required point be (x,o) as it lies on X-axis and then use the condition of equidistance.

Exercise 8.2

Q5. Hint: The area of the triangle formed by the three points (x,y), (3,4) and (-5,-6) is zero.

Exercise 8.3

Q4. Hint: Use the formula of the centroid (x,y) of a triangle having (x_1,y_1), (x_2,y_2) and (x_3, y_3) as vertices

$$x = \frac{x_1+x_2+x_3}{3} \quad , \quad y = \frac{y_1+ y_2 + y_3}{3}$$

Exercise 8.5

Q2. Hint: Distance of any point (x,y) from x-axis is y.

Information About Availability of Teaching Aids

Geoboard with rubber bands helps in performing simple experiments to secure some understanding of the simple coordinate geometry of lines and polygons. It is interesting to know that Prof. G Polya used Geoboard for demonstrating Pythagoras Theorem.

Project Work

Fixing the coordinates of the vertices of a square to which the axes are (i) the axies of symmetry and (ii) No axes of symmetry.

5. Chapter Test

a. Oral Test

i) What are the distances of the points (3,-6), (0,5) and (6,0) from the x-axis ?

ii) Tell the distances of the points (-7,-5), (6,0) and (0,4) from the y-axis.

iii) Which of the points (2,3), (-1,5), (0,4), (6,0), (2,-5), (0,-3), (-6,0) lie on the x-axis ?

iv) Identify the points that lie on the y-axis:

(-4, 0), (5,-7), (0,3), (-11,3), (0,-4).

v) A and B are respectively the points:

a) (11,4) and (4,11)

b) (-3,0) and (-3,5)

c) (5,-7) and (5,7)

d) (0,8) and (0,-8)

In which case, is AB parallel to y-axis ? Identify the use where AB passes through the origin.

vi) If P and Q are respectively the points.

a) (3, -4) and (-3, -4)

b) (7, 0) and (-5,0)

c) (-11,9) and (4,9)

d) (-1,2) and (1,-2)

tell, in each case, whether PQ is parallel to x-axis. In which case, does PQ pass through the origin ?

vii) Given that (i) A=(5,2), B=(-3,2), C=(-6,-1) and D=(-6,1);

ii) A = (0,3), B=(5,3), C=(-1,3), D=(1,3).

Tell whether AB is parallel or perpendicular to CD.

viii) What is the distance of the point (3,4) from the origin?

ix) Give the mid-point of the line joining origin and (-6,-2).

x) What is the mid point of the line segment joining (5,-8) and (1,-2) ?

xi) For which lines their slopes are not defined ?

xii) What is the distance between the points (-7,1) and (5,6)?

xiii) Find if the angle of inclination of the line joining (3,-1) and (-5,2) to the x-axis is acute or obtuse.

b. <u>Written Test</u>

i) Find if the join of (-5,1) and (3,-6) is parallel to the join of (4,-2) and (12,-9).

ii) Show that the points (1,4), (3,-2) and (-3,16) are collin

iii) One end point of a segment is (2,4). A point $(\frac{13}{4}, \frac{7}{8})$ divides the segment in the ratio of 5:3.

Find the other end point. (Ans: (4, -1)

6. <u>Additional Reading Materials For Enrichment</u>

i) <u>Translation of axes:</u>

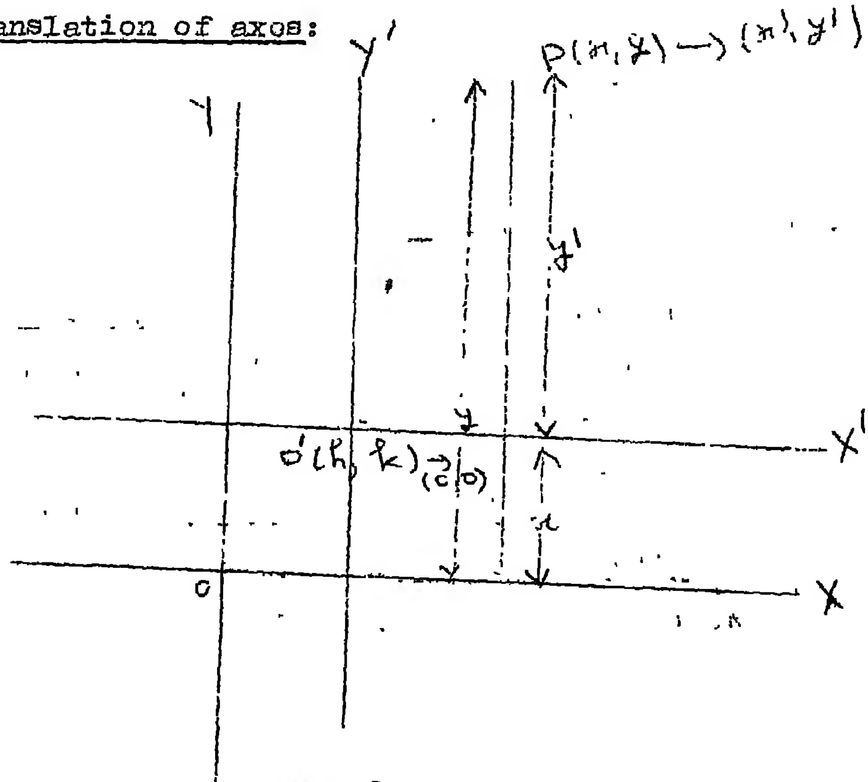


Fig. 8.6

sometimes shifting the origin to a particular point and taking axes of reference parallel to the original ones, the equations of a certain class of curves reduce to elegant and standard forms which enable the students to solve these problems easily. It is important to observe that when the axes undergo such sort of translation, distance between two points remains invariant

Let the translation be given by

$x \longrightarrow x' + h$
$y \longrightarrow y' + k$

Then $P(x_1,y_1)$ becomes $P(x_1-h, y_1-k)$ and $Q(x_2,y_2)$ becomes $Q(x_2-h, y_2-k)$.

$$\text{Now } PQ = \sqrt{(\overline{x_2-h} - \overline{x_1-h})^2 + (\overline{y_2-k} - \overline{y_1-k})^2}$$

$$= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

ii) <u>Suggested References</u>

1. Kells, L.M. and Stots, H.C. <u>Analytic Geometry</u> New York, Prentice Hall (Earlier introduction of translation and rotation of Analytic Geometry).

2. Blen, G.A. <u>Mathematical Interpretation of Geometrical and Physical Phenomena</u>; American Mathematical Monthly 40 (1933) : 472-480.

CHAPTER - 9

STRAIGHT LINE

1. Introduction

Having been exposed to the concepts of distance between two points, slope of the line joining two points and the algebraic processes in chapter 8, the student has the readiness to study the equation of a straight line in particular and general forms. He has seen how, given two points, any other point on (or outside) the line joining the two points can be found. The development and use of this technique shows the style of thinking in Analytic Geometry.

It would be necessary to cover briefly the topic 'determinants', as it is required in this chapter. A hand-out may be given to the students for preparatory study.

Here we may recall the elimination method of solving two linear equations of the form

$$a_1x + b_1y + c_1 = 0,$$
$$a_2x + b_2y + c_2 = 0,$$

Solving, we get $x = \dfrac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}$; $y = \dfrac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$

Provided $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$

Which can be combined as

$$\frac{x}{b_1c_2-b_2c_1} = \frac{y}{c_1a_2-c_2a_1} = \frac{1}{a_1b_2-a_2b_1}$$

This can be rewritten in determinant form as

$$\frac{x}{\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & c_2 \\ a_1 & a_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}$$

A cross shows that to evaluate a determinant of order 2, we have to multiply diagonally and an arrow heading towards right shows that the sign is positive while that heading towards left is taken negative.

Similarly, a determinant of order 3 can be evaluated. The students can be taught the rule expansion of a general determinant of order 3,

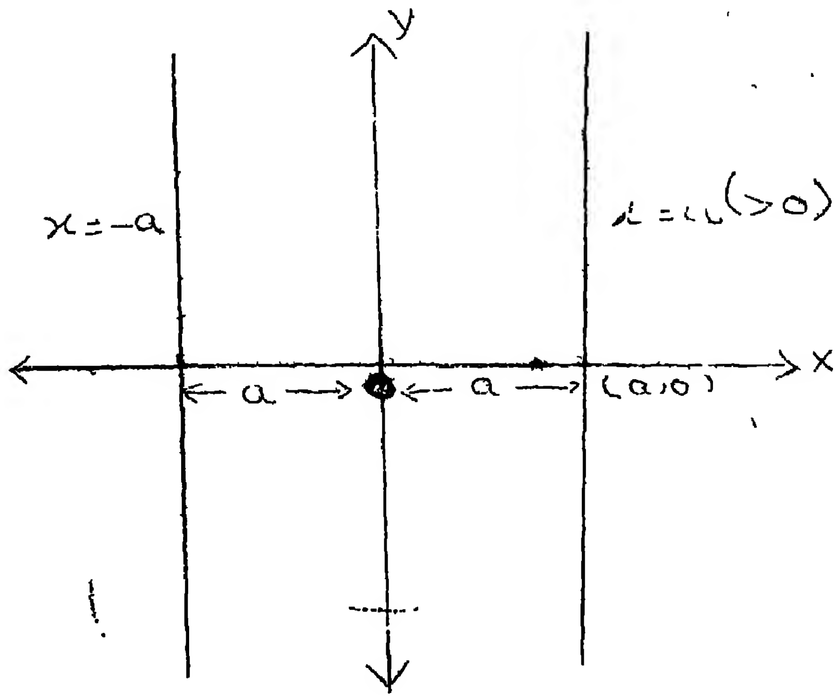
e.g.
$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = a\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \; b\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} \; c\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

2. <u>Content Analysis</u>

In this section, the number of each sub section is in accordance with that in the text book.

9.1. <u>To find the equation of a straight line Parallel to an Axis:</u>

The students known that every point on a line parallel to Y-axis has a fixed abscissa and hence its equation is $x=a$ where a is the abscissa of any point on this line. This abscissa may not necessarily be positive. If there is a line on which every point has abscissa $-a$, $(a > 0)$, its equation will be $x = -a$.



Similarly, the ordinate of every point on a line parallel to X-axis may also be positive or negative and hence the equation of the line will be of the form $y=\pm b$ $(b > 0)$.

In continuation of the chapter 1 of the text book, we can define the x-axis, y-axis, lines parallel to y axis and lines parallel to X-axis in the form of sets.

1) <u>X-axis</u>

Its equation is $y = 0$.

We get it from $ax+by+c = 0$.

setting $a = 0$, $b=1$, $c=0$.

The X-axis is, therefore,

$\{(x,y) \mid ax+by+c+0, \quad a=c=0, \ b=1\}$

ii) Y-axis

Its equation is $x=0$.

So $a=1$, $b=0$, $C=0$

The y-axis is, therefore

$\{(x,y) \mid ax+by+C=0,\ a=1,\ b=C=0.\}$

iii) A line parallel to y-axis

Its equation is of the form $x=h$

S $a=1$, $b=0$, $C=-h$.

The line is

$\{(x,y) \mid ax+by+C=0,\ a=1,\ b=0,\ C=-h.\}$

iv) A line parallel to X-axis

Its equation is of the form $y=k$

So $a=0, b=1$, $C=-k$.

The line is

$\{(x,y) \mid ax+by+C=0,\ a=0,\ b=1,\ C=-k.\}$

9.2 The Point-slope form

Let the slope of the line be m. If it passes through a point (x_1,y_1), then the line is $\{(x,y) \mid y-y_1=m(x-x_1)\} = \{(x,y) \mid mx-y+(-mx_1+y_1)=0\}$. This set may also represent x-axis or a line parallel to x-axis but not y-axis nor a line parallel to y-axis as the slopes of y-axis and of a line parallel to y-axis are undefined.

9.3 Two-Point Form

The students know that the required form is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1); \quad x_1 \neq x_2$$

where (x_1,y_1) and (x_2,y_2) are the two given points on the line.

Here, an important property of a line can now be given as the following theorem

<u>Theorem</u>: If the line $ax+by+c = 0$ contains the points (x_1,y_1) and (x_2,y_2) then it contains every point given by $(\lambda x_1 + \mu x_2, \lambda y_1 + \mu y_2)$ where $\lambda + \mu = 1$. (Recall the section Formula).

<u>Proof</u>: Since (x_1,y_1) and (x_2,y_2) are points on the line

$ax+by+c=0$

$ax_1+by_1+c=0$

$ax_2+by_2+c=0$

Therefore $\lambda(ax_1+by_1+c) + \mu(ax_2+by_2+C) = 0$

$\Rightarrow a(\lambda x_1+\mu x_2) + b(\lambda y_1+\mu y_2) + C = 0 \; [\because \lambda + \mu = 1$

Hence the theorem

<u>Conversely</u>,

If a locus contains $(\lambda x_1+ \mu x_2, \lambda y_1 + \mu y_2)$, where $\lambda, \mu \in R$ and $\lambda + \mu = 1$, whenever it contains (x_1,y_1) and (x_2, y_2), the locus is a line.

<u>Proof</u>:

Let (x,y) be a point on the locus.

Then $x = \lambda x_1 + \mu x_2$ (1).

and $y = \lambda y_1 + \mu y_2$ (2)

Eliminating μ from (1) and (2), we get

$$xy_2 - x_2y = \lambda(x_1y_2 - x_2y_1) \quad \ldots (3)$$

and eliminating λ from (1) and (2), we get

$$xy_1 - x_1y = -\mu(x_1y_2 - x_2y_1) \quad (4)$$

Equations (3) and (4) give

$$(y_2-y_1)x - (x_2-x_1)y - (x_1y_2 - x_2y_1) = 0 \quad (\because \lambda + \mu = 1)$$

which is a linear equation in x and y.

Hence the locus is a line.

The two-point form can also be represented in the set form as

$$\left\{(x,y) \;\middle|\; y = y_1 + \frac{x - x_1}{x_2 - x_1} + (y_2-y_1)\right\}$$

$$\Rightarrow \left\{(x,y) \;\middle|\; (y_2-y_1)x + (x_1-x_2)y + (x_2y_1 - x_1y_2) = 0.\right\}$$

9.5 Slope-Intercept Form

If a line is not parallel to y-axis, it will meet the Y-axis at some point. The ordinate of this point is called the Y-intercept of the line. Obviously, if C is the Y-intercept the point (0,c) lies on the line. Hence we substitute (0,C) for (x_1, y_1) in $y-y_1=m(x-x_1)$ and get $y-c = m(x-0)$

or, $y = mx+C$.

Here m is the slope of the line.

Now, any line parallel to this line will have the same slope although it may have a different Y-intercept.

Therefore, the equation

$y = mx + \lambda,\ \lambda \in R$

represents the set (or family) of all those lines which are parallel to the line $y = mx + c$.

9.8 General Form

The students know that the general form of the equattion of a straight line is $Ax + By + C = 0$

This equation can be reduced to the normal form as

$$-\frac{A}{\sqrt{A^2 + B^2}}\,x - \frac{B}{\sqrt{A^2 + B^2}}\,y = \frac{C}{\sqrt{A^2 + B^2}}$$

This holds good when $C \geq 0$.

If $C < 0$, then

$$\frac{A}{\sqrt{A^2 + B^2}}\,x + \frac{B}{\sqrt{A^2 + B^2}}\,y = \frac{-C}{\sqrt{A^2 + B^2}}$$

is the required form. An emphasis is to be given that in the normal form of the equation of a straight line

$X \cos W + y \sin W = p,$

the right hand side term p is the length of perpendicular drawn to the line from the origin, which, being a measurement, is always a non-negative term. Therefore, the right hand side of the transformed equation has to be a non-negative term.

3. Learning Outcomes

a. Essential Learning Outcomes For All

The essential competence, a student is expected to achieve by reading this chapter includes the ability

i) to write the equation of X-axis, Y-axis, a line parallel to Y-axis and line parallel to x-axis.

ii) to find the equation of a line if one point on the line and the slope of the line is given. He should be able to write the equation of the line even if its slope is not given but another line is given with which it is parallel or perpendicular.

iii) to write the equation of a line on which two points are given.

iv) to write the equation of a line whose intercepts are given.

v) to find the equation of a line whose

a) slope and Y-axis intercept are given,

b) distance from origin and the angle between the perpendicular drawn through origin to the line and positive side of X-axis are given.

vi) to find angle between two given lines.

vii) to verify concurrency of three lines.

viii) to prove some of the theorems of geometry using coordinate geometry (analytical method).

ix) to find the distance of a given point from a given line.

x) to identify parallel and perpendicular lines.

b) <u>Learning Outcomes For the Higher Group</u>

I) To be able to find the intercepts of a line if its equation in any form is given.

ii) Competency in transforming the general equation of a line into perpendicular form.

iii) Ability to verify whether a given pair of lines is a pair of parallel lines, perpendicular lines, coincident lines or none of these.

iv) Ability to find the distance between two parallel lines.

v) Ability to identify from a pair of lines, the line which is farther from (or nearer to) the origin.

vik) Ability fo find the new coordinates of different points after translation of axes and to be able to verify that distance between two points and area of a triangle etc. remain invariant after translation of axes.

vii) Ability to write the equations of lines whose distances from origin and the angles that the lines make with the positive side of x-axis are given.

4. <u>Teaching Strategies</u>

<u>Motivation</u>

An able teacher knows that his lessons will have greater impact through the process of motivation. This motivation can be induced through different sorts of activities. Learning by doing and examples from our daily life etc. make our lessons interesting. So, to motivate the students, following sort of activities are suggested.

i) Draw a line l in xy-plane. Ask the students to spell out the various characteristics that can be noticed about the ~~x~~ line with regard to the axes.

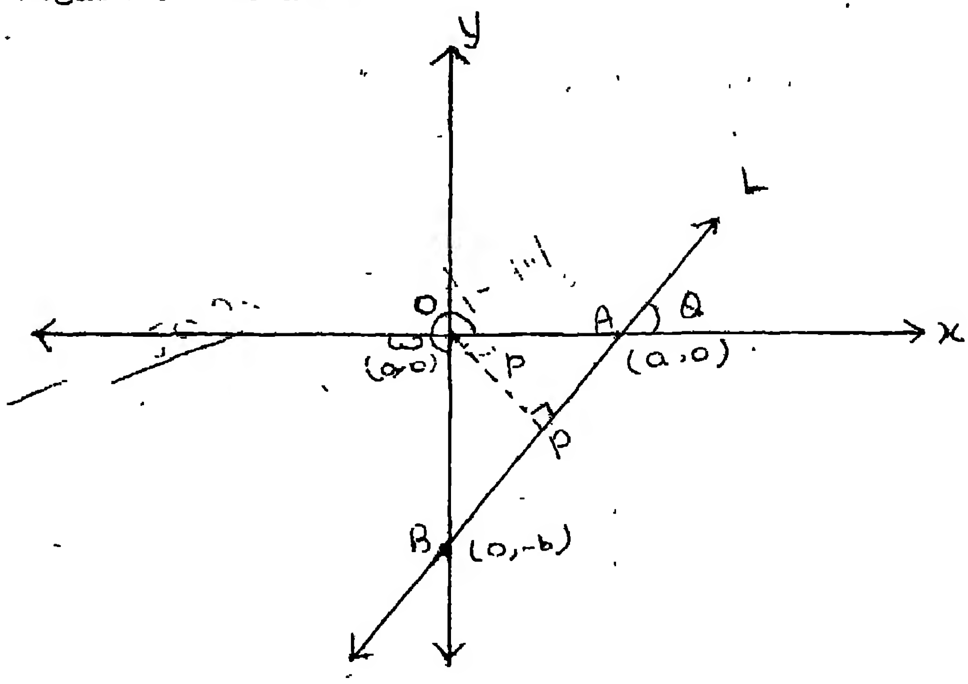


Fig. 9.2

Some of these characteristics are

a. The points A and B lie on the line. These points are the points of intersection of the line with the axes.

b. The distances of those points from the origin are called the measurement of intercepts of the line with the axes. Here the intercepts are a and -b respectively.

c. The inclination of the line is Q.

d. Through the two points A and B, we get the unique line l.

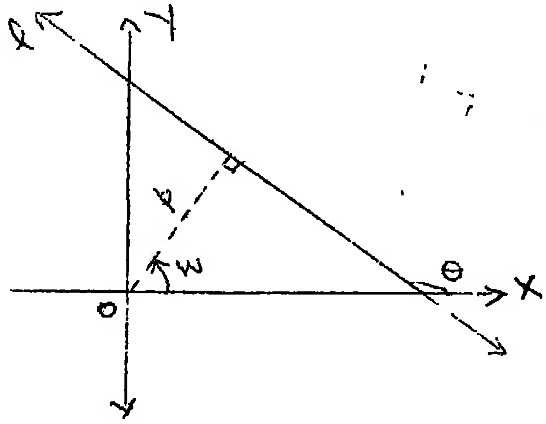
e. The slope of the line l is tan Q.

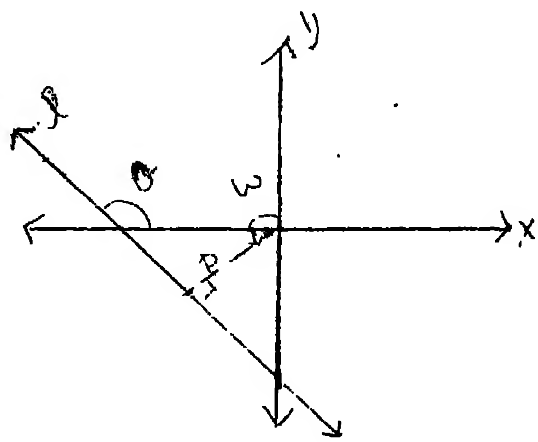
f. The distance of the line from the origin is p.

g. The angle between the positive side of x-axis and the perpendicular drawn through origin to the line is W, and not the acute angle, $\angle AOP$, as the angle is measured anticlockwise taking the positive side of X-axis as the initial line.

h. There is a relation between θ and W, $W = 270^\circ + \theta$.

Here, the students may be asked if they can draw some lines where this relation does not hold. Some of the students will immediately draw figures as shown below:





$W = \theta - 90^\circ$

Fig. 9.3

$W = 90^\circ + \theta$

Fig. 9.4

This preliminary activity creates in the students readiness to find the equation of a line in different forms.

ii)

Rail

Road

Fig. 9.5

There is a road parallel to a railway track, upto some distance as in the figure. Both of them are along straight lines at a distance of 10 metres. What is the slope of the angle between them ?

Most of the students will come out with the right answer, 0. Now, keeping one of them as on axis of reference (say X-axis), what will be the equation of the other ?

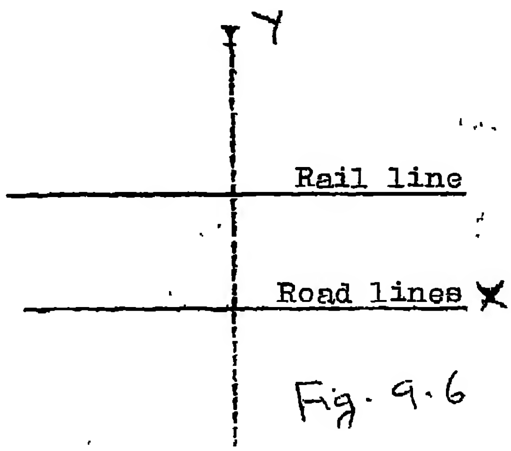


Fig. 9.6

The equation of rail-line is y = 10.

Keeping Rail-line as X-axis,

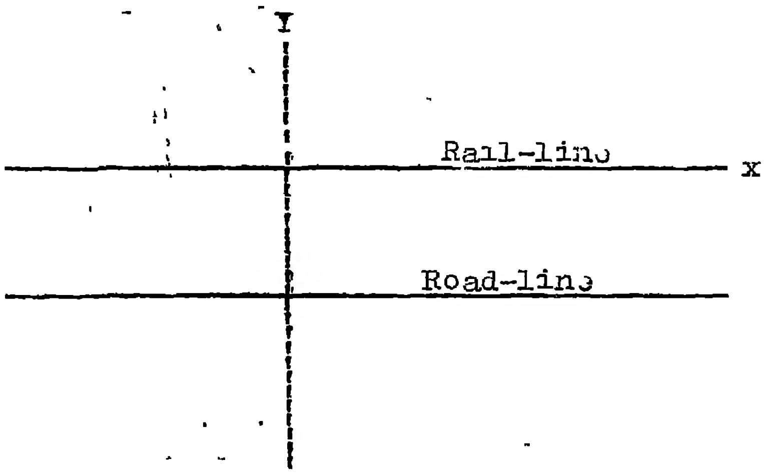


Fig. 9.7

The equation of road line is y= -10.

iii) In a school the students of each class stand in a separate 'line' for the morning assembly. Parallel lines are formed in succession starting from the lowest class. One day the instructor asks them to first take two steps ahead and then one step to their left. Suppose the step of each student is uniform. What is the distance between the 3rd and 7th students of the line of class XI ? More precisely, will it change now ? What about distance between 9th students of class XI and XII ? What about the distance between 5th student of class XI and 8th student of class X ?

With this example, we can start to the expose them with translation of axes.

<u>Misconceptions</u>

i) Be certain to distinguish between a line whose equation is $x = 0$ and the X-axis. Similarly the lines $y=0$ and the Y-axis are absolutely different lines.

ii) The slope of a line is a measure of its steepness. If a line rises sharply to the right (in the anticlockwise sense), the slope is large (a positive value). If a line rises slowly to the right, the slope is small (a positive value). But a line rising sharply to the left (in the clockwise sense) has a small slope (a negative value) while a line rising slowly to the left has comparatively a larger slope (although a negative value).

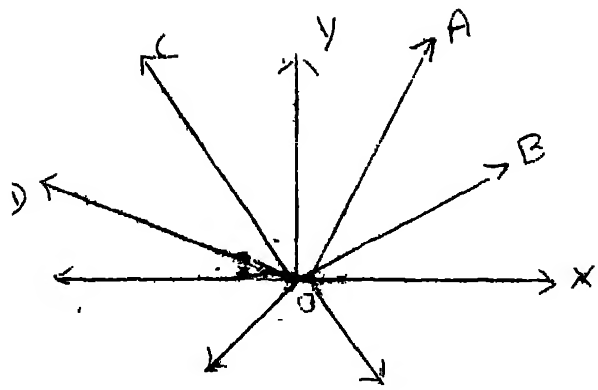


Fig. 9.8

here slope of OC < slope of OD < Slope of OB < slope of OA

iii) The slope of a line perpendicular to X-axis does not exist, as the slope m which is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$ exists only when $x_2 \neq x_1$. Make sure to convince the students that a line which has slope 0 is absolutely different from the line which does not have slope.

Additional Exercises

i) Find the equation of the line joining the points A(2,-5) and B(1,3). Also find the equation of another line which is perpendicular to it and passes through the midpoint of AB.

(Ans. $8x+y-11=0$, $2x-16y-19=0$).

ii) Find the equation of the line passing through the points (-1, 3) and (6,1). Also find the length of the segment of this line intercepted between the axes.

(Ans: $2x+7y-19=0$, 9.9 (approx.)

iii) The vertices of the ABC are A(-4,5), B(-1,0) and C(2,3). AD is one of its medians. Find the equation of the perpendicular drawn from B to AD (Ans. 9x - 7y + 9=0).

iv) Find the ratio in which the line segment joining the points (1,-3) and (5,4) is divided by the line joining the points (-2,-3) and (4,0). (Ans. 6:7)

v) Find the equation(s) of line(s) whose distance from origin is 4 and makes an angle of 120° with the positive side of x-axis.

(Ans: $\sqrt{3}x+y=\pm 4$).

vi) The equations of the sides of a triangle are x+2y=0, 4x+3y=5 and 3x+y=0. Find the coordinates of the orthocentre of the triangle. (Ans: $\frac{4}{5}, -\frac{3}{5}$)

<u>Hints For Difficult Problems</u>

<u>Exercise 9</u>

Q6. Inclinations of the two lines are 30° & 60°. Also inclination of $y = x+3$ is 45°. $\therefore$ $y=x+3$ bisects the angle between two given lines, since all the three lines pass through (0,3).

Q8. Use one point-slope form as (-3,0) is a point on the line. The slope, m_2, of the line is given by $m_2 = -\frac{1}{m_1}$, where m_1 is the slope of $3x+5y=4$.

Q9. Let x-intercept be a, then y-intercept is 9-a. Therefore the equation is $\frac{x}{a} + \frac{y}{9-a} = 1$. The point (2,2) lies on it.

$\therefore \frac{2}{a} + \frac{2}{9-a} = 1$

Solving a = 3 or 6.

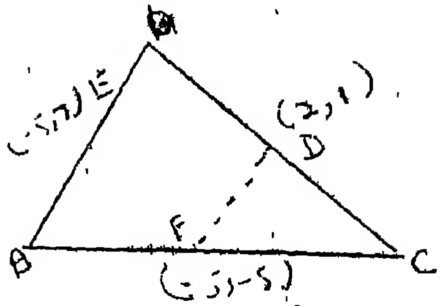


Fig 99

Q.11 For equation of AB

slope of AB = slope of FD

Use one-point-slope form.

Similar is the procedure for BC and CA.

Q.15 Find the distance of (0,0) from $\frac{x}{a} + \frac{y}{b} = 1$

$$\frac{\frac{1}{a} \cdot 0 + \frac{1}{b} \cdot 0 - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = p$$

Q.18 Transform the two equations into normal forms and compare the lengths of perpendiculars from origin.

Q.22 Use $\tan Q = \pm \frac{m_2 - m_1}{1+m_1 m_2}$ or use the formulae(F_1)&(F_2) given in the next section, (ii)

Q.28 $y=4 \Rightarrow 0.x+1.y-4=0$ therefore distance from (2,3),

$$d = \left|\frac{0.2 + 1.3-4}{\sqrt{0^2 + 1^2}}\right| = 1$$

<u>Proofs of some theorems</u>

<u>Distance of A point From A Line</u>

i) Without using the normal form, the distance from a point to a line can be found as explained below in two ways:

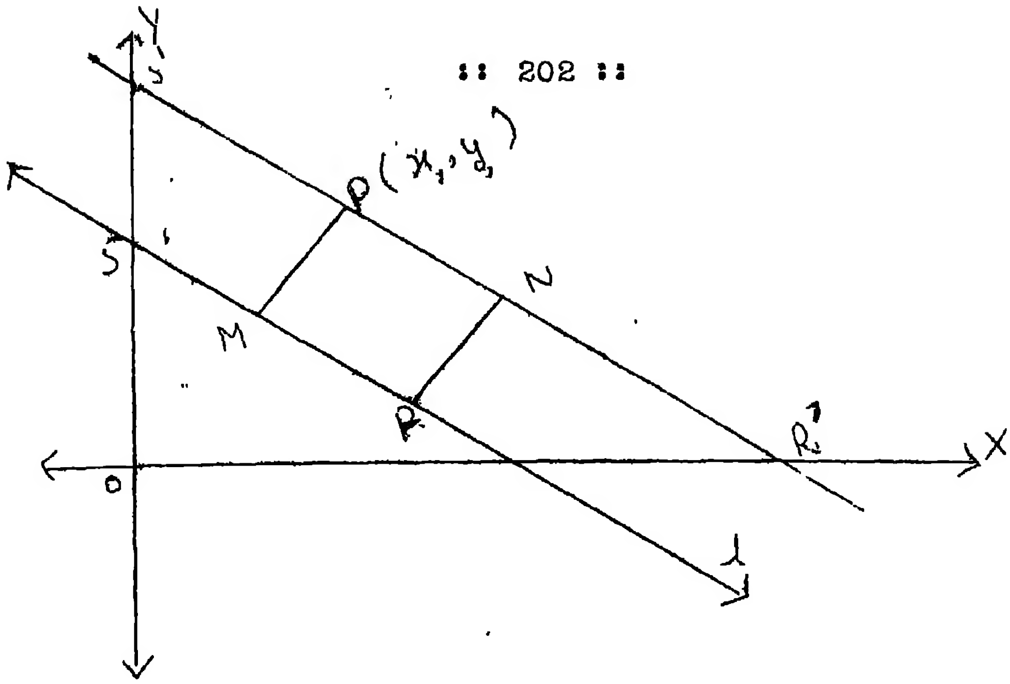


Fig. 9.10

Let l be the line

$\{(x,y) \mid ax+by+c = 0\}$

and $P(x_1, y_1)$ be a point outside the line.

The line through P & Parallel to l is $ax+by+k=0$ where $k=-ax_1-by_1$

Now $d = PM = RN$

and $\frac{RN}{RR^1} = \frac{OS}{SR}$ $\quad [\because \triangle RNR^1 \sim \triangle OSR]$

$\therefore d = RN = \frac{RR^1 . OS}{SR}$

But $RR^1 = OR^1 - OR = \frac{-k}{a} + \frac{c}{a} = \frac{c-k}{a}$

$OS = -\frac{c}{b}$

& $SR = \sqrt{(-\frac{c}{b})^2 + (\frac{c}{a})^2} = \pm \frac{c}{ab}\sqrt{a^2+b^2}$

$\therefore d = \left| \frac{c-k}{a}\left(-\frac{c}{b}\right)\left(\pm \frac{ab}{c\sqrt{a^2+b^2}}\right) \right|$

$\frac{|c-k|}{\sqrt{a^2+b^2}} = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

Note: If the equation of the line is taken as $y=mx+c$, then the line through P Parallel to the given line is $y=mx+c^1$. From the figure $d=(c^1-c)\cos\alpha = \frac{c^1-c}{\sqrt{1+m^2}}$, where $\tan\alpha = m$

Alternatively.

Consider the following figure:

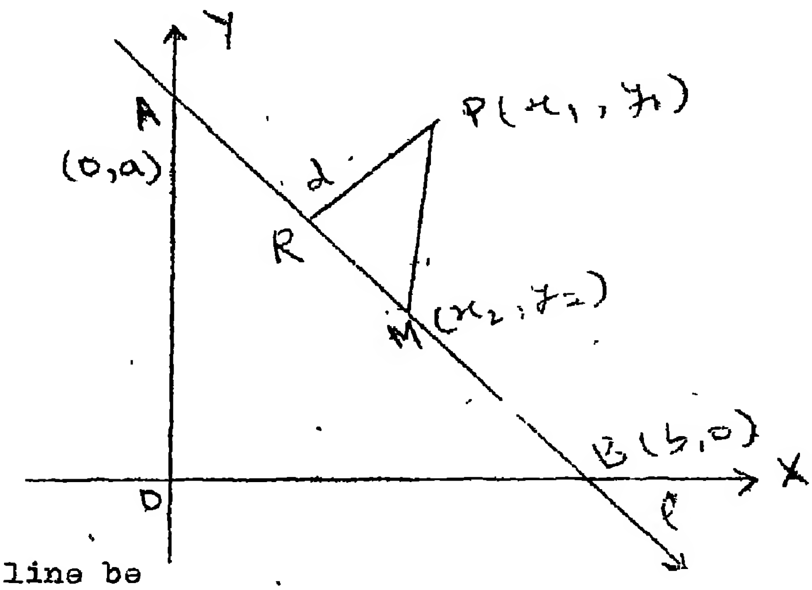


Fig. 9.11

Let the line be

$\{(x,y) \mid ax + by + c = 0.\}$

Consider PM in the perpendicular direction to the x-axis, meeting the line at M. Now, M is (x_2, y_2) and $x_2=x_1$.

$$MP = y_1 - y_2$$

$$= y_1 - \left(\frac{-ax_1 - c}{b}\right)$$

$$= \frac{ax_1 + by_1 + c)}{b}$$

$$\frac{MP}{RP} = \frac{\sqrt{a^2 + b^2}}{|b|} \qquad (\because \triangle RPM \sim \triangle OBA)$$

$$\therefore d = \frac{|b||(ax_1 + by_1 + c)|}{|b|\sqrt{a^2 + b^2}}$$

$$= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

ii) <u>Angle Between Two Lines</u>

Let the two lines be

$a_1x + b_1y + c_1 = 0$ (1)

& $a_2x + b_2y + c_2 = 0$ (2)

The slope, m_1, of the line (1) is $-\frac{a_1}{b_1}$

and the slope, m_2, of the line (2) is $-\frac{a_2}{b_2}$

If Q is the angle between the lines $y=m_1x+c_1$ and $y=m_2x+c_2$,

then it has been proved in the text book that

$$\tan Q = \frac{m_2 - m_1}{1+m_1m_2}$$

Substituting the values of m_1 & m_2 in

$$\tan Q = \frac{a_1b_2 - a_2b_1}{a_1a_2 + b_1b_2} \quad (F_1)$$

When the angle is $(\pi - Q)$ then it is given by

$$-\tan Q = \frac{a_1b_2 - a_2b_1}{a_1a_2 + b_1b_2}$$

$$\tan Q = \frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2} \quad (F_2)$$

Importance of this form can be seen in doing problems 21 & 22 of the Exercise 9 of the text book.

<u>Problem</u>: Find the equation(s) of the line(s) through the origin making an angle of 60° with the line $x + y\sqrt{3} + 3\sqrt{3} = 0$.

Solution: The line passes through origin.

Let $a_1x + b_1y = 0$ be the required equation.

Using $\tan Q = \dfrac{a_1b_2 - a_2b_1}{a_1a_2 + b_1b_2}$,

we get $\tan 60^o = \dfrac{a_1\sqrt{3} - b_1}{a_1 + \sqrt{3}\, b_1}$

$$\Rightarrow \sqrt{3} = \frac{a_1\sqrt{3} - b_1}{a_1 + \sqrt{3}\, b_1}$$

$$\Rightarrow \sqrt{3}\, a_1 + 3b_1 = \sqrt{3}a_1 - b_1$$

$$\Rightarrow 4b_1 = 0$$

$$\Rightarrow b_1 = 0$$

Hence one of the lines is $a_1x = 0$ or $x = 0$.

Using $\tan Q = \dfrac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}$,

We get $\sqrt{3} = \dfrac{b_1 - \sqrt{3}\, a_1}{a_1 + \sqrt{3}\, b_1}$

$$\Rightarrow \sqrt{3}\, a_1 + 3b_1 = b_1 - \sqrt{3}\, a_1$$

$$\Rightarrow 2\sqrt{3}\, a_1 = -2b_1$$

$$\text{or } a_1 = -\frac{b_1}{\sqrt{3}}$$

Hence, the other line is

$$-\frac{b_1}{\sqrt{3}}x + b_1y = 0$$

or $x = \sqrt{3}\, y$.

iii) An Alternative Proof of Theorem 9.2

Every straight line has an equation of the form $Ax+By+C=0$ Where A, B and C are constants

Proof:- Let (x_1, y_1) and (x_2, y_2) be two points on

$$Ax + By + C = 0. \qquad (1)$$

$$\left.\begin{aligned} \text{Then, } Ax_1 + By_1 + C &= 0 \\ Ax_2 + By_2 + C &= 0 \end{aligned}\right\}$$

$$\Rightarrow A(x_1-x_2) + B(y_1-y_2) = 0 \ldots\ldots\ldots(2)$$

Since A, B are constants, there exists a k, such that $A = k(y_2-y_1)$ and $B = k(x_1-x_2)$.

These values of A and B clearly satisfy (2)

So (1) can be written as $x(y_2-y_1) + y(x_1-x_2)+C=a\ldots\ldots(3)$

But the standard form of the equation of the line joining two points is given by

$$(x_2-x_1)(y-y_1) = (y_2-y_1)(x-x_1)$$

$$\Rightarrow (y_2-y_1)x - (x_2-x_1)y - \left\{(y_2-y_1)x_1 - (x_2-x_1)y_1\right\} = 0. \text{----} (A)$$

Comparing(3) with (4), we find ; (1) and hence (1) is a straight line.

Teaching Aids

Visual teaching aids like overlaying transparencies showing the ddiagrams developed step by step and exhibiting several cases on an over head projector (OHP) - may be used by a teacher. If an OHP is available, this approach will

save time of a teacher and generate interest among the students.

Project Work

i) Find the ratio in which the segment joining (x_1, y_1) and (x_2, y_2) is cut by the line $ax+by+C=0$, not passing through (x_1, y_1) and (x_2, y_2).

(Ans. $\dfrac{|ax_1+by_1+C|}{|ax_2+by_2+C|}$

ii) In (i), examine the case when the ratio is 1:1

iii) Find the slope of a line given in parametric form.

$$\{(x,y) \mid x = a+bk,\ y=c+dk,\ k \in R\}$$

5. Chapter Tests

a. Oral Test

i) What is the equation of a line with inclination of 45° and passing through $(0,-3)$?

ii) Find the intercepts of the line $5x-8y+3=0$.

iii) What is the equation of the line perpendicular to $5x-8y+3=0$ and passing through origin ?

iv) What is the distance of the line $3x+4y-30=0$ from the origin?

v) What is the distance between the point $(2,3)$ and the line $y=4$?

vi) What is the equation of a line whose distance from the origin is 3 units and perpendicular to the line from the origin makes an angle of 30° with the positive side of x-axis ?

vii) Transform the equation $4x-\frac{3}{2}y-1=0$ to normal and intercept forms.

b. Written Test

i) In what ratio is the line joining (2,1) and (5,2) divided by the line joining (3,2) and (5,0) ? [Ans 1:1]

ii) Find the slope form of the equation of a line from its parametric form

$x = x_1 + \gamma \cos\theta$ and $y = y_1 + \gamma \sin\theta$.

Ans: $y-y_1=\tan\theta\,(x-x_1)$ or $y-y_1=m(x-x_1)$

iii) Find the equation of the line passing through the points (a,c) and $(a+b,\ c+d)$. (Ans: $dx-by+(bc-da)=0$

iv) Prove analytically that the diagonals of a rhombus are perpendicular to each other.

v) Prove analytically that if the diagonals of a parallelogram are equal, then the parallelogram is a rectangle.

vi) Find the direction of the line which contains $A(1, \sqrt{3})$ such that its point of intersection with the line $x+\sqrt{3}\,y-8=0$ is at a distance of 2 units from the point A. (Ans: $\sqrt{3}\,x-y=0$).

vii) Find the equations of the medians of a triangle whose vertices are $(-5,2)$, $(4,-6)$ and $(1,7)$. Also, show that they are concurrent.

(Ans: $x+5y-5=0$, $7x+4y-4=0$ & $6x-y+1=0$)

viii) For what value of a will the three lines $3x+y+2=0$, $2x-y+3=0$ and $x+ay-3=0$ be concurrent ? (Ans: 4)

ix) Find the coordinates of the circumcentre of the triangle with vertices (4,3), (-2,3) and (6,-1).

(Ans: (1,-1).

6. <u>Additional Reading Material</u>

Ballantine, J.P. & Jerbert, A.P. "Distance from a line or plane to a point", American Mathematical Monthly 59 (1952), 242-244.

CHAPTER 10

FAMILY OF LINES

1. Introduction

In the preceding two chapters, points and straight lines have been considered.

Given two points, the distance of the segment joining them and slope of the line passing through have been algebraically explained. Any point of a straight line passing through two given points can be obtained in parametric form.

Analytical methods of proving the geometrical properties have been introduced. Emphasis has been laid on the fact that geometric properties are independent of the choice of origin. By suitably shifting the origin and translating axes algebraic calculations get reduced and simplified in many problems.

Equation of a line in general and various particular forms have been vividly explained and usefulness of each form has been pointed out to get the required characteristic of a line.

In this chapter the family of straight lines passing through the point of intersection of a pair of straight lines along with the family of lines passing through a particular point, parallel and perpendicular to a given straight line will be explained. The condition for representing a pair of straight lines by the general equation of second degree will be investigated. Moreover, emphasis

will be laid on the **homogeneous** equation of second degree, which in general, represents a pair of lines passing through the origin.

2. Content Analysis

In this section the number of each subsection is in accordance with the textbook.

10.1 Equation of Family of Lines

Obviously, sufficiently large number of lines can be drawn from a given point P (x_1, y_1). These all form a family of lines passing through P. This family of lines is represented by the equation, $y - y_1 = m(x - x_1)$, where $m \in R$, is an arbitrary constant.

Similarly, sufficiently large number of lines can be drawn parallel to a given line $Ax + By + c = 0$. The equation $Ax + By + \lambda = 0$ represents the family of lines, which are parallel to $Ax + By + c = 0$, as the slopes of lines represented by both the equations are equal to $-\frac{A}{B}$.

Likewise sufficiently large number of lines perpendicular to the given line $Ax + By + c = 0$, can be drawn. These two form a family of lines. The family of such lines is represented by the equation.

$Bx - Ay + \mu = 0$, where $\mu \in R$ is an arbitrary constant. The slope of the line represented by the equation $Bx - Ay + \mu = 0$, is

$\frac{B}{A}$, whereas the slope of line represented by the equation $Ax + By + c = 0$ is $-\frac{A}{B}$.

Since $-\frac{A}{B} \cdot \frac{B}{A} = -1$, hence the equation $Bx - Ay + \mu = 0$ represents a family of lines perpendicular to the given line $Ax + By + c = 0 \ \forall \mu \in R$.

Evidently, sufficiently large number of lines can pass through the point of intersection of a pair of nonparallel lines.

$$A_1x + B_1y + C_1 = 0$$

and $$A_2x + B_2y + C_2 = 0.$$

The equation of this family of lines is represented by either of the following two equations.

$$A_1x + B_1y + C_1 + \lambda (A_2x + B_2y + C_2) = 0$$

and $A_2x + B_2y + C_2 + \mu (A_1x + B_1y + C_1) = 0$

where $\lambda, \mu \in R$ are arbitrary constants.

10.2 Pair of Straight Lines Through Origin

Consider the equation

$$xy = 0.$$

Evidently, it implies that $x = 0$ and/or $y = 0$. As $x = 0$ and $y = 0$ are respectively the equations of the axis of Y and axes of X, and hence the equation $xy = 0$ represents a pair of lines passing through origin. Similarly the equation $x^2 - y^2 = 0$ implies that

$x - y = 0$ and/or $x + y = 0$ which are the equations of a pair of lines passing through origin. Again $y - m_1x = 0$ and $y - m_2x = 0$ are two separate equations of a pair of lines passing through the origin. The combined equation of this pair of lines is

$$(y - m_1x)(y - m_2x) = 0$$

or $y^2 - (m_1 + m_2)xy + m_1m_2x^2 = 0$, which too is a homogeneous equation of second degree as the sum of the exponents of x and y in each term is 2. The most general form of a homogeneous equation of second degree is

$$ax^2 + 2hxy + by^2 = 0$$

The general homogeneous equation of second degree $ax^2 + 2hxy + by^2 = 0$ too represents a pair of lines passing through origin.

Solving $ax^2 + 2hxy + by^2 = 0$ as a quadratic equation for x, we have

$$x = \frac{-2hy \pm 2\sqrt{h^2y^2 - aby^2}}{2a}$$

$$= \left(\frac{-h \pm \sqrt{h^2 - ab}}{a}\right) y$$

or $ax = -hy \pm \sqrt{h^2 - ab}\, y$.

Therefore $ax^2 + 2hxy + by^2 = 0$.

$$\Longrightarrow (ax + hy + \sqrt{h^2 - ab}\, y)(ax + hy - \sqrt{h^2 - ab}\, y) = 0$$

which implies that

$ax + (h + \sqrt{h^2 - ab})y = 0$ and/or $ax + (h - \sqrt{h^2 - ab})y = 0$.

These two equations separately represent a pair of lines passing through origin if $h^2 - ab \geqslant 0$. These lines are real and distinct, if $h^2 > ab$, coincident if $h^2 = ab$, and the lines do not exist if $h^2 < ab$.

10.3 Angle Between The Pair of Lines

We know that $y = m_1 x$ and $y = m_2 x$ are two separate equations of a pair of lines passing through the origin. Hence combined equation of this pair of lines is

$$(y - m_1 x)(y - m_2 x) = 0$$

or $$y^2 - (m_1 + m_2) xy + m_1 m_2 x^2 = 0.$$

We have already proved that

$ax^2 + 2hxy + by^2 = 0$ represents a pair of lines passing through origin. Hence on comparing the co-efficients of like terms we have :

$$\frac{m_1 m_2}{a} = \frac{-(m_1 + m_2)}{2h} = \frac{1}{b}$$

$$\therefore \quad m_1 m_2 = \frac{a}{b} \text{ and } m_1 + m_2 = \frac{-2h}{b}$$

If θ is the angle between pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ then $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$

$$= \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}$$

$$= \frac{2\sqrt{\frac{h^2}{b^2} - \frac{a}{b}}}{1 + \frac{a}{b}}$$

$$= \frac{2\sqrt{h^2 - ab}}{a + b}$$

Hence $\theta = \operatorname{Tan}^{-1}\left[\frac{2\sqrt{h^2 - ab}}{a + b}\right]$

Evidently, the condition for coincidence of the lines is

$$h^2 = ab.$$

In other words, for the coincidence of the lines, the expression $ax^2 + 2\,hxy + by^2$ should be a perfect square.

Condition for perpendicularity is

$$a + b = 0$$

Consequently, $ax^2 + 2\,hxy - ay^2 = 0$ represents a pair of mutually perpendicular lines passing through origin.

10.4 Equation Of The Bisectors Of The Angles

The equation of the bisectors of the angles between the pair of lines $ax^2 + 2hxy + by^2 = 0$ is —

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

or $\quad x^2 - y^2 = \frac{a-b}{h} xy$, which represents a pair of mutually perpendicular lines passing through origin, as the sum of the co-efficients of x^2 and y^2 is equal to zero.

10.5 Condition For The General Equation Of Second Degree to Represent Two Straight Lines

We know that $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are separate equations of a pair of lines. On combining these two as follows we get $(A_1x + B_1y + C_1)\ (A_2 x + B_2y + C_2) = 0$

or $\quad A_1A_2x^2 + xy\ (A_1B_2 + B_1A_2) + B_1B_2y^2 + x\ (A_1C_2 + A_2C_1) + y\ (B_1C_2 + B_2C_1) + C_1C_2 = 0.$

Obviously, it is of the following form $ax^2 + 2\ hxy + by^2 + 2\ gx + 2\ fy + c = 0 \quad \ldots (A)$

This equation (A) is known as the general equation of second degree. Actually in three unknowns x, y, and z, the homogeneous equation of second degree is $ax^2 + by^2 + Cz^2 + 2\ fyz + 2\ gxz + 2\ hxy = 0$

Putting $z = 1$, it assumes the form

$$ax^2 + by^2 + c + 2\ fy + 2\ gx + 2\ hxy = 0$$

which is exactly the same as equation (A).

Thus we have seen that the general equation of second degree represents a pair of lines, if the expression $ax^2 + 2\ hxy + b\ y^2 + 2\ gx + 2\ fy + c$ can be factorized in two linear factors of the form $A_1x + B_1y + C_1$ and $A_2x + B_2y + C_2$.

The condition for the general equation of second degree
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of lines may be obtained in various ways. Some are given below to enable the readers to appreciate their relative merits.

The expression on left hand side of (A)

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$, on multiplying by a assumes the form $a^2x^2 + 2ax(hy + g) + aby^2 + 2afy + ca$

$$= (ax + hy + g)^2 - (hy + g)^2 + aby^2 + 2afy + ca$$

$$= (ax + hy + g)^2 - y^2(h^2 - ab) - 2y(hg - af) - (g^2 - ac)$$

$$= (ax + hy + g)^2 - \left\{y^2(h^2 - ab) + 2y(hg - af) + (g^2 - ac)\right\}.$$

Which can be factorized in two linear factors in x and y, provided $y^2(h^2 - ab) + 2y(hg - af) + (g^2 - ac)$ is a perfect square. The condition for which is

$$(hg - af)^2 - (h^2 - ab)(g^2 - ac) = 0$$

or $\quad \cancel{h^2g^2} + a^2f^2 - 2hgaf - \cancel{h^2g^2} + ach^2 + abg^2 - a^2bc = 0$

or $\quad af^2 - 2fgh + ch^2 + bg^2 - abc = 0$

or $\quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

which is the required condition.

Alternative Method.

Let (x_1, y_1) be the point of intersection of the pair of lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

Now translating the origin to the point (x_1, y_1) and taking a new parallel system of axes the transformed form of the equation (A) is

$$a(x+x_1)^2 + 2h(x+x_1)(y+y_1) + b(y+y_1)^2$$
$$+ 2g(x+x_1) + 2f(y+y_1) + c = 0$$

or
$$ax^2 + 2hxy + by^2 + 2x(ax_1 + hy_1 + g)$$
$$+ 2y(hx_1 + by_1 + f) + ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1$$
$$+ 2fy_1 + c = 0 \quad \ldots\ldots\ldots \quad (B)$$

As equation (B) represents a pair of lines passing through origin it must be a homogeneous equation of second degree. Consequently, coefficient of x, y and constants will vanish separately.

Hence

$$ax_1 + hy_1 + g = 0 \quad \ldots\ldots \quad (1)$$
$$hx_1 + by_1 + f = 0 \quad \ldots\ldots \quad (2)$$

and $ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \ldots \quad (3)$

Multiplying (1) and (2) respectively by x_1 and y_1 and adding we have :

$$ax_1^2 + 2hx_1y_1 + gx_1 + fy_1 + by_1^2 = 0 \quad \ldots\ldots \quad (4)$$

Subtracting (4) from (3) we have —

$$gx_1 + fy_1 + c = 0 \quad \ldots\ldots \quad (5)$$

Eliminating $x_1 + y_1$ from (1), (2) and (5) we have —

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

on simplifying we obtain

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, which is the desired condition.

3. LEARNING OUTCOMES

(a) Essential Learning Outcomes For All.

After comprehensive study of the content of this chapter each and every student of the class must be able to :

(i) write the equation of the family of lines passing through a given point, parallel/perpendicular to a given line or passing through the point of intersection of a pair of given lines.

(ii) Recognise whether a given equation of second degree is homogeneous or not.

(iii) State that a homogeneous equation of second degree represents a pair of lines passing through origin.

(iv) Test the perpendicularity/coincidence of the pair of lines represented by a given homogeneous equation of second degree.

(v) Find out separate equations of the pair of lines represented by a given equation of second degree and vice-versa.

(vi) Find the included angle between a pair of lines represented by a given equation, without finding the separate equations of the lines.

(b) Learning Outcomes For the Higher Groups

Higher ability group of students must acquire the following competencies :

(i) Competence to check, whether the given equation of second degree represents a pair of lines or not. If in affirmative, he/she must be able to obtain the separate equations of the pair of lines represented by it.

(ii) Competence to find out the included angle between the given pair of lines and consequently must be able to check the perpendicularity or parallelism of a given pair of lines.

Teaching Strategies

Motivation

An experienced teacher adopts his own way for motivating the students. Here attempt is made to suggest some of the ways.

Activity (1) The students may be asked to take a point $P(x_1, y_1)$ and draw a number of lines passing through it. How may lines passing through P, can be drawn ?

Sufficiently large number of lines can pass through a given point. These all lines form a family of lines passing through a given point.

What is the equation of the family of lines passing through (x_1, y_1) ? From the point-slope form of the equation of a straight line, the answer is

$$\frac{y - y_1}{x - x_1} = m$$

or $\quad y - y_1 = m\,(x - x_1)\ \forall\ m \in R.$

Thus the equation $y - y_1 = m\,(x - x_1)$ represents a family of lines passing through (x_1, y_1).

<u>Activity-2</u> Ask the students to draw the line $y = x$ and again prompt other students to draw parallels to it and passing through each of the points $(0,1)$, $(0,2)$, $(0,3)$ ..... and $(0,+1)$, $(0,-2)$, $(0,-3)$ ...... What are the equations of the lines thus drawn ?

The equations of the lines drawn are $y = x-1$, $y = x-2$, $y = x-3$ ...... and $y = x+1$, $y = x+2$, $y = x+3$ ......
What is the equation of a line parallel to $y = x$ and passing through the point $(0,\lambda)$?

The answer is $y = x-\lambda$, which represents a family of lines parallel to $y = x$, where $\lambda \in R$ is an arbitrary constant.

Activity-3 Ask the students to draw a perpendicular to the given line $y = x$ passing through origin.

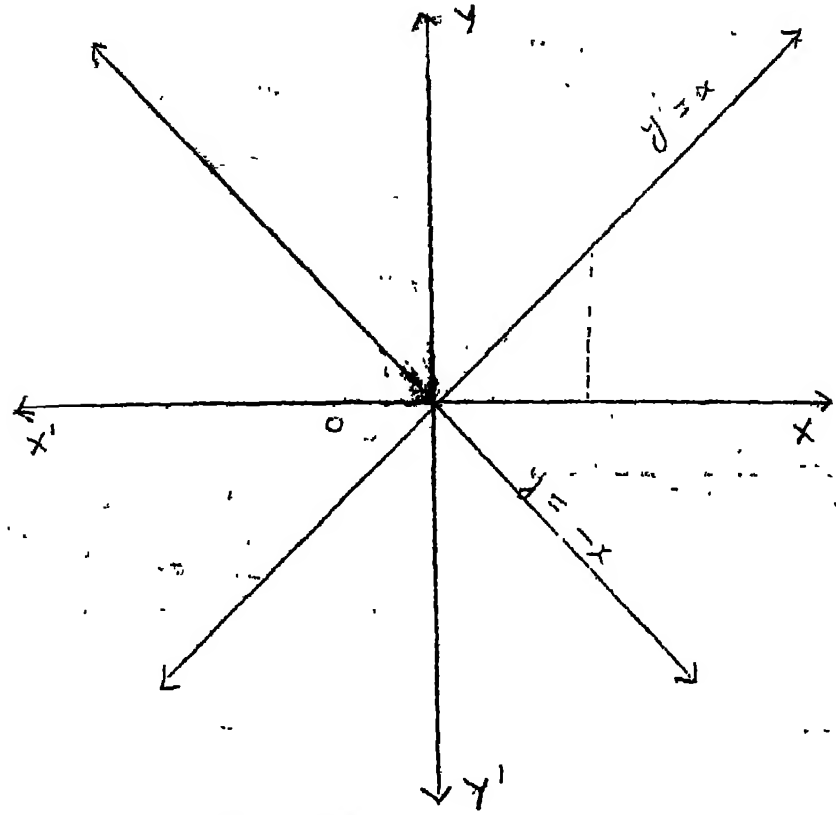


Fig. 10.1

What is the equation of the line thus drawn ?

The equation of the line thus drawn is $y = -x$.

Other students of the class must be prompted to draw perpendiculars to the line $y = x$ and passing through various points $(0, n)$ where $n \in N$. What are the equations of the lines thus drawn ?

The answer is $y = -x + n$ where $n \in N$. Thus $y = -x + n$ represents a family of lines perpendicular to the line $y = x$.

After proper motivation, family of lines passing through the point of intersection of the pair of lines $Ax + By + C = 0$ and $A'x + B'y + C' = 0$ should be introduced.

Prompt the students to solve $ax^2 + 2hxy + by^2 = 0$ as a quadratic equation in x and y separately and observe the independence of the results obtained in both the ways.

Misconceptions/Common Errors.

Following common errors should be emphasised to the students:

While solving numerical problems based on the general equation of second degree, students commit mistakes in writing the values of a, b, c, f, g and h with proper signs. It should be noted that the values of f, g and h are actually half of the coefficients of y, x and xy respectively. For instance in the equation

$$6x^2 - xy - 12y^2 - 8x + 29y - 14 = 0$$

or $$6x^2 + 2\left(-\tfrac{1}{2}\right)xy - 12y^2 + 2(-4)x + 2\left(\tfrac{29}{2}\right)y - 14 = 0$$

$\therefore$ $h = -\frac{1}{2}$; not -1 ; $g = -4$, not -8 ;

$f = \frac{29}{2}$, not 29 ; $a = 6$ and $b = -12$

Additional Exercises

(i) Find the product of the lengths of perpendiculars drawn from (x_1, y_1) to the pair of lines $ax^2 + 2hxy + by^2 = 0$.

(ii) Find the area of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my = n$.

(iii) Find the equation to the pair of lines which are perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ and pass through the origin.

(iv) Find the condition for representing a pair of parallel lines by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and find the distance between this pair of lines.

Solutions/Hints For Difficult Problems

Exercise 10.1

Q.No.4 – Family of lines passing through the point of intersection of the given lines is $(2x + 3y - 4) + \lambda(x - 5y + 7) = 0$

or $x(2 + \lambda) + y(3 - 5\lambda) + (7\lambda - 4) = 0$

or $$\frac{x}{\frac{4 - 7\lambda}{2 + \lambda}} + \frac{y}{\frac{4 - 7\lambda}{3 - 5\lambda}} = 1$$

For required line x intercept $= -4$

or $$\frac{4 - 7\lambda}{2 + \lambda} = -4$$

or $4 - 7\lambda = -8 - 4\lambda$

or $12 = 3\lambda$

$\therefore$ $\lambda = 4$

Hence the required equation is

$6x - 17y + 24 = 0.$

Exercise 10.2

Q.No. 5

Equation of the pair of lines assumes the form ,

$$x^2 (\cos^2\alpha - \sin^2\alpha) - 2\, xy \cos\alpha \sin\alpha = 0$$

or $\quad x \left[x \cos 2\alpha - y \sin 2\alpha \right] = 0$

or $\quad x (y - \cot 2\alpha\ x) = 0$

Separate equations of the pair of lines are $x = 0$ and $y = \tan (\pi/2 - 2\alpha)\, x$.

Evidently, $x = 0$ is the axis of y.

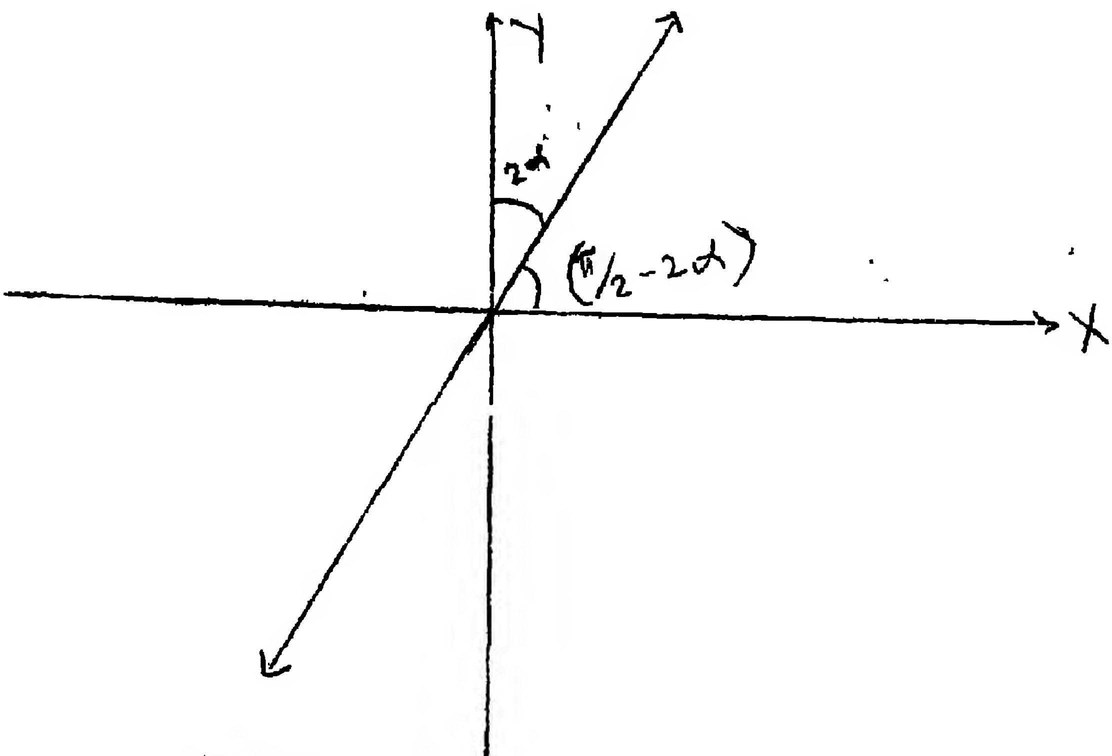


Fig. 10.2

∴ The included angle is 2α.

Exercise 10.3

Q.No. 5

Let the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a pair of lines intersecting at (x^1, y^1).

Translating the origin to (x^1, y^1) the transformed form of the given equation is

$$a(x + x^1)^2 + 2h(x + x^1)(y + y^1) + b(y + y^1)^2 + 2g(x + x^1)$$
$$+ 2f(y + y^1) + c = 0$$

or
$$ax^2 + 2hxy + by^2 + 2x(ax_1 + hy_1 + g) + 2y(hx_1 + by_1 + f)$$
$$+ ax_1^2 + by_1^2 + 2hx_1y_1 + 2gx_1 + 2fy_1 + c = 0$$

As it represents a pair of lines passing through the origin and hence it will be a homogeneous equation of second degree and thus co-efficient of x, y and constant term will separately vanish.

Therefore

$$ax_1 + hy_1 + g = 0$$

$$hx_1 + by_1 + f = 0$$

Solving these equations, we have

$$\frac{x_1}{hf - bg} = \frac{y_1}{gh - af} = \frac{1}{ab - h^2}.$$

Consequently

$$x_1 = \frac{fh - bg}{ab - h^2}$$

and
$$y_1 = \frac{gh - ab}{ab - h^2}$$

5. **Chapter Tests**

(a) **Oral Test**

(i) Find separate equations of the lines represented by $x^2 - 36y^2 = 0$.

Ans: $(x + 6y) = 0$ & $x - 6y = 0$.

(ii) Find the angle between pair of lines represented by $x^2 + 8xy - y^2 = 0$ Ans: 90°

(iii) Find the angle between pair of lines $4x^2 - 4xy + y^2 = 0$. Ans: 0°.

(b) **Written Test**

1. Show that the four lines given by the equation
$3x^2 + 8xy - 3y^2 = 0$
and $3x^2 + 8xy - 3y^2 + 2x - 4y - 1 = 0$
form a square. Find the equations of the diagonals of the square.

II. Prove that the equation $bx^2 - 2hxy + ay^2 = 0$ represents a pair of lines which are at right angles to the pair of lines given by the equation $ax^2 + 2hxy + by^2 = 0$.

III. Find the condition for the slope of one of the lines represented by $ax^2 + 2hxy + by^2 = 0$, may be λ times. The slope of the other.

Hint: Let m and λm be the slope of the lines represented by $ax^2 + 2hxy + by^2 = 0$. $m + \lambda m = -\frac{2h}{b}$ and $\lambda m^2 = \frac{a}{b}$. Eliminate λ.

Reference: As given in the Chapter 8.

CHAPTER 11

CIRCLE AND FAMILY OF CIRCLES

1. Introduction

By now students must have learnt, to find the distance between two points. The knowledge thus obtained will form the basis for finding the equation of a circle in this chapter and the equation of the parabola, the ellipse and the hyperbola in the next chapter.

Since circle has been studied in High School classes by synthetic method. With a few exceptions, nothing new in the field of the geometry of circle will be obtained here. On the other hand, an attempt has been made to introduce the algebraic processes that will be used not only in this chapter but also in the next chapter.

2. Content Analysis

In this section the number of each section is in accordance with the textbook.

Circle, parabola, ellipse and hyperbola are curves of intersection by a plane with a right circular double cone. To be specific enough, if the measure of the angle between the intersecting plane and the axis of the cone is equal to the semi-vertical angle of the cone, the resulting curve of intersection is an ellipse and if the measure of the angle between the intersecting plane and axis of the cone is less than the semi-vertical angle of the cone, the intersecting curve turns

out to be a hyperbela. When the intersecting plane is perpendicular to the axis of the cone, the intersecting curve is a circle. As the circle, the parabela, the ellipse and the hyperbela are the sections of a cone and hence are called conic-sections or conics.

As the distance formula is to be employed in obtaining the equation of each of these curves, the curve is to be looked upon as a locus of points satisfying appropriate distance properly between points on it and on its axes.

11.1 Standard Form of the Equation of a Circle

The circle is defined to be a locus of a point, whose distance from a fixed point (h, k) is a positive constant r. It means circle is a set of points $\left\{(x, y) \,\middle|\, (x-h)^2 + (y-k)^2 = r^2\right\}$. Thus the equation of a circle is $(x-h)^2 + (y-k)^2 = r^2$. The distance of each point of this circle from the fixed point (h, k) is a positive constant r. The fixed point (h, k) is called the centre of the circle, and positive constant which is the measure of the distance of each and every point of the circle from the centre, is called the radius of the circle. Taking origin as the centre of the circle, the equation of the circle assumes the form —

$$(x-0)^2 + (y-0)^2 = r^2$$

or $\quad x^2 + y^2 = r^2$, which is an elegant and simple standard form of the equation of a circle.

11.2 General Form of the Equation of a Circle

The Equation of Circle is :

$$(x - h)^2 + (y - k)^2 = r^2$$

on expansion of the terms we have

$$x^2 + y^2 - 2hx - 2ky + (k^2 + h^2 - r^2) = 0$$

or $$x^2 + y^2 + (-2h)\,x + (-2k)\,y + (h^2 + k^2 - r^2) = 0$$

This leads to the conclusion that the general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots\ldots \qquad (A)$$

Since this assumes the form

$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c$$

Obviously the centre of the circle represented by the equation (A) is $(-g, -f)$ and the radius is $\sqrt{(g^2 + f^2 - c)}$.

The general equation of the second degree is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (B)$$

On comparison of (A) and (B) we observe,

(i) $h = 0$, i.e. coefficient of xy in the general equation of a circle is 0.

(ii) $a = b = 1$ i.e. coefficients of x and y in the general equation of a circle are equal to 1.

From above it is evident that the equation

$$ax^2 + ay^2 + 2gx + 2fy + c = 0 \quad \ldots\ldots \qquad (C)$$

will also represent a circle.

The equation (C) assumes the forms

$$x^2 + y^2 + 2\frac{g}{a}x + 2\frac{f}{a}y + \frac{c}{a} = 0$$

The centre of the circle (C) is

$\left(-\frac{g}{a}, -\frac{f}{a}\right)$ and radius is $\sqrt{\left(\frac{g^2}{a^2} + \frac{f^2}{a^2} - \frac{ac}{a^2}\right)}$

$$= \frac{1}{a}\sqrt{(g^2 + f^2 - ac)}$$

From above it is evident that the circle represented by the equation (A) will exist if $g^2 + f^2 - C > 0$, and the circle will not exist if $g^2 + f^2 - C < 0$. In the case, when $g^2 + f^2 - C = 0$, the equation (A) will represent a circle of zero radius i.e. a Point Circle

11.3 Equation of a Circle in Parametric Form

Consider the circle $x^2 + y^2 = r^2$. It is easy to observe that the point $(r\cos\alpha, r\sin\alpha)$ always exists on the circle $x^2 + y^2 = r^2$.

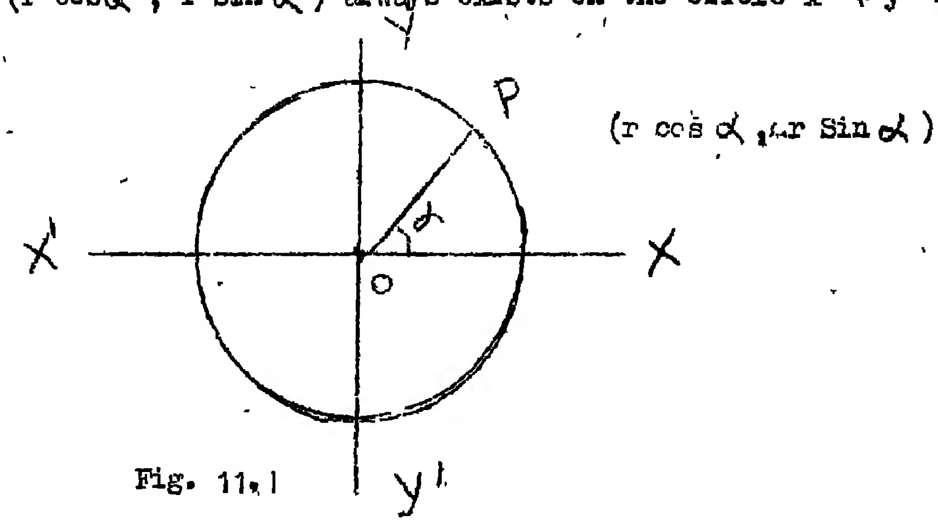


Fig. 11.1

on assigning different values to ranging from 0 to 2π, we get different point of the circle $x^2 + y^2 = r^2$. Thus we can say that

$x = r \cos\theta$ and $y = r \sin\theta$ represent any point on the circle, $x^2 + y^2 = r^2$, where θ is the measure of any angle such that $0 \leq \theta \leq 2\pi$

Obviously the parametric form of the equation $(x - h)^2 + (y - k)^2 = r^2$ is $x = h + r\cos\theta$ and $y = k + r\sin\theta$, in terms of a parametre.

11.4 Equation of a Circle when the End Points of a Diametre are Given

The end points of a diametre of a circle are (x_1, y_1) and (x_2, y_2). Different ways can be employed to obtain the equation of the circle.

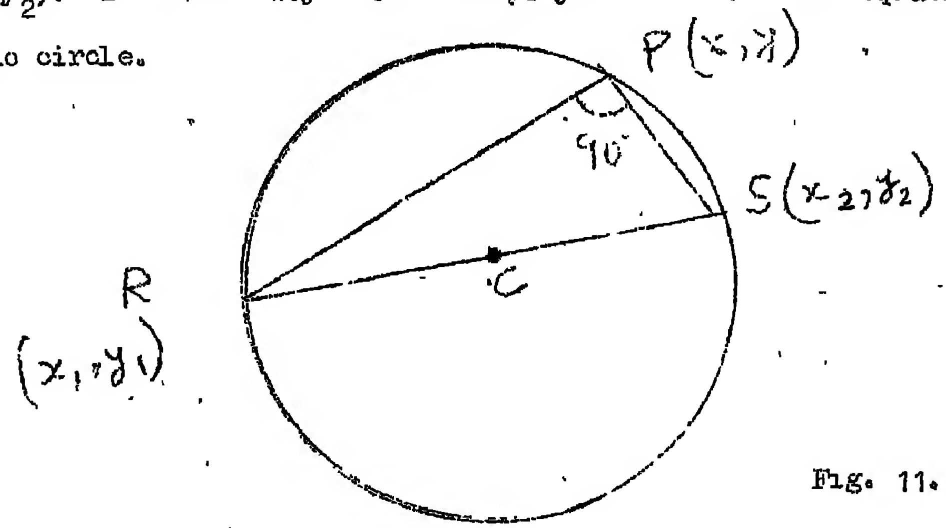


Fig. 11.2

From previous knowledge of geometry students are expected to know that $\angle RPS = 90^\circ$, where P is any point on the circle having $R(x_1, y_1)$ and $S(x_2, y_2)$, the ends of a diametre R S. Obviously RP and PS are mutually at right angles and hence

$$(\text{Slope of PR}) \times (\text{Slope of PS}) = -1$$

or $$\left(\frac{y - y_1}{x - x_1}\right) \times \left(\frac{y - y_2}{x - x_2}\right) = -1$$

or $\quad (y - y_1)\ (y - y_2) = -\ (x - x_1)\ (x - x_2)$

or $\quad (x - x_1)\ (x - x_2) + (y - y_1)\ (y - y_2) = 0$

which is required equation of the circle.

The students should be prompted to employ other methods to obtain the equation of the circle and verify that every method leads to the same result.

11.5 Points of Intersection of a Line and a Circle with Centre at Origin. Condition of Tangency.

We know that the circle having orign as its centre and r as its radius is $x^2 + y^2 = r^2$. ..... (i)

The equation of a line is $y = mx + c$ ...... (ii)

For finding the points of intersection of (i) and (ii) solving them as simultaneous equations, we get :

$$x^2 + (mx + c)^2 = r^2$$

or $\quad x^2\ (1 + m^2) + 2\ mcx + (c^2 - r^2) = 0$ ..... (iii)

which is a quadratic equation in x and hence it will have two roots. Both of the roots will be real and distinct if

$$\cancel{4}\, m^2c^2 - \cancel{4}\,(1 + m^2)\ (c^2 - r^2) > 0$$

or $\quad -c^2 + r^2\ (1 + m^2) > 0$

or $\quad c^2 < r^2\ (1 + m^2)$

Both the roots of (iii) will be real and coincident if

$$c^2 = r^2\ (1 + m^2).$$

again both the roots of (iii) will not exist if

$$c^2 > r^2 (1 + m^2)$$

Consequently, it follows that the line (ii) will intersect the circle (i) in two distinct points, in one point or in no real point according as

$$r > \frac{c}{\sqrt{1 + m^2}}$$

$$= \frac{c}{\sqrt{1 + m^2}}$$

$$< \frac{c}{\sqrt{1 + m^2}}$$

<u>Condition of Tangency</u> :

Evidently the line (ii) will be a tangent to the circle (i) if $c = \pm r \sqrt{1 + m^2}$. Thus $y = mx \pm r\sqrt{1 + m^2}$ represents a pair of parallel tangents of the circle $x^2 + y^2 = r^2$.

11.6 Equation of a Tangent to a Circle and Length of the Tangent.

Consider the circle $x^2 + y^2 = r^2$. For obtaining the equation of the tangent of the circle at a point (x_1, y_1) we can employ the following knowledge of geometry.

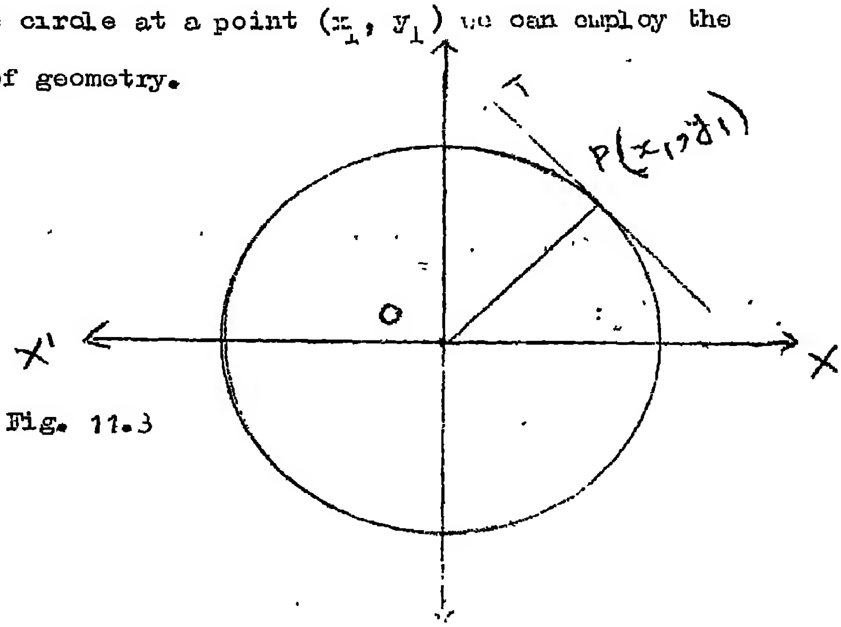


Fig. 11.3

The radius passing through the point of contact of the tangent of a circle is perpendicular to the tangent.

Evidently the tangent PT at P (x_1, y_1) of the circle $x^2 + y^2 = r^2$ is perpendicular to the radius OP.

$$\text{Slope of OP} = \frac{y_1 - 0}{x_1 - 0} = \frac{y_1}{x_1}$$

Let (x, y) be any other point of the tangent PT.

$$\therefore \quad \text{Slope of PT} = \frac{y - y_1}{x - x_1}$$

$$\because \quad (\text{Slope of OP}) \times (\text{Slope of PT}) = -1$$

$$\text{or} \quad \frac{y_1}{x_1} \times \left(\frac{y - y_1}{x - x_1}\right) = -1$$

or $\quad yy_1 - y_1^2 = -xx_1 + x_1^2$

or $\quad yy_1 + xx_1 = x_1^2 + y_1^2$

or $\quad xx_1 + yy_1 = r^2$, as (x_1, y_1) lies on the circle $x^2 + y^2 = r^2$.

Thus $xx_1 + yy_1 = r^2$ is the equation of the tangent to the circle $x^2 + y^2 = r^2$, at a point (x_1, y_1) of the circle.

In the case of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, having a point (x_1, y_1) on it, we can write the equation of the tangent at (x_1, y_1).

Translating the axes at the point $(-g, -f)$ the transformed form of the equation

$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c$$

will be

$$X^2 + Y^2 = h^2 + f^2 - c$$

Let $(x_1, y_1) \longrightarrow (X_1, Y_1)$

Equation of the tangent at (X_1, Y_1), a point of the Circle $X^2 + Y^2 = g^2 + f^2 - C$ is

$$XX_1 + YY_1 = g^2 + f^2 - c.$$

This in accordance with the original system of axes transforms into

$$(x + g)\ (x_1 + g) + (y + g)\ (y_1 + f) = g^2 + f^2 - c$$

or $\quad xx_1 + yy_1 + g\ (x + x_1) + f\ (y + y_1) + c = 0$

which is the equation of the tangent at the point (x_1, y_1).

Length of the tangent

Consider the circle $x^2 + y^2 = r^2$ and a point $P\ (x_1, y_1)$ outside the circle. Let PT be the tangent on the circle from P and T is the point of contact of the tangent. The measure of the line-segment PT is called the length of the tangent on the circle $x^2 + y^2 = r^2$ from $P\ (x_1, y_1)$.

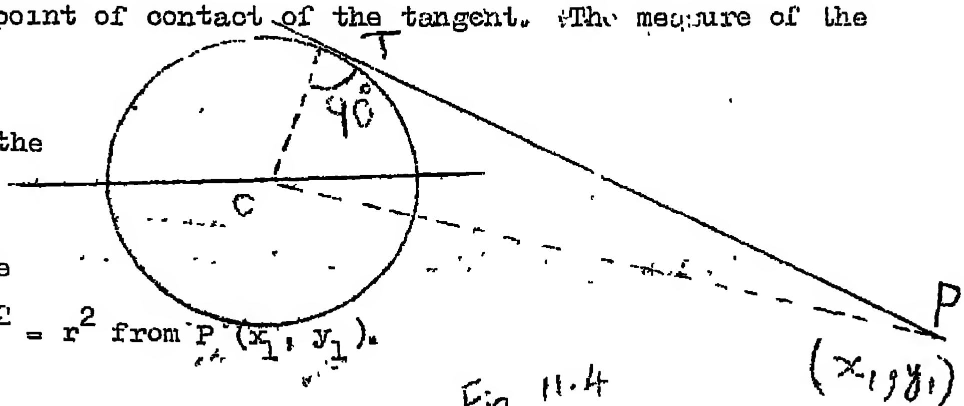


Fig 11.4

From geometry $\angle OTP = 90^\circ$.

Applying Pythagoras Theorem we have

$$PT^2 = OP^2 - r^2$$

$$= x_1^2 + y_1^2 - r^2$$

$\therefore$ $PT = \sqrt{x_1^2 + y_1^2 - r^2}$, which is the length of the tangent from P (x_1, y_1) on the circle $x^2 + y^2 = r^2$.

Similarly, the length of tangent from the point (x_1, y_1) to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ will be

$$= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

11.7 Families of Circles Through the Intersection of Two Circles.

By now, students must have attained the competence to write down the coordinates of the centre of a circle represented by an equation : Ask the students to find the centre of the circle

$$x^2 + y^2 - 6x - 8y + \lambda = 0$$

The centre of the circle is (3, 4).

Again ask the students to report the radius of the circle.

Radius of the circle $= \sqrt{3^2 + 4^2 - \lambda}$

$$= \sqrt{25 - \lambda}$$

Assigning different values to λ, we get a concentric set of circles having (3,4) as its centre. The set of concentric circles having (3,4) as centre form a family, and thus equation $x^2 + y^2 - 6x - 8y + \lambda = 0$ represents a family of concentric circles having (3,4) as centre.

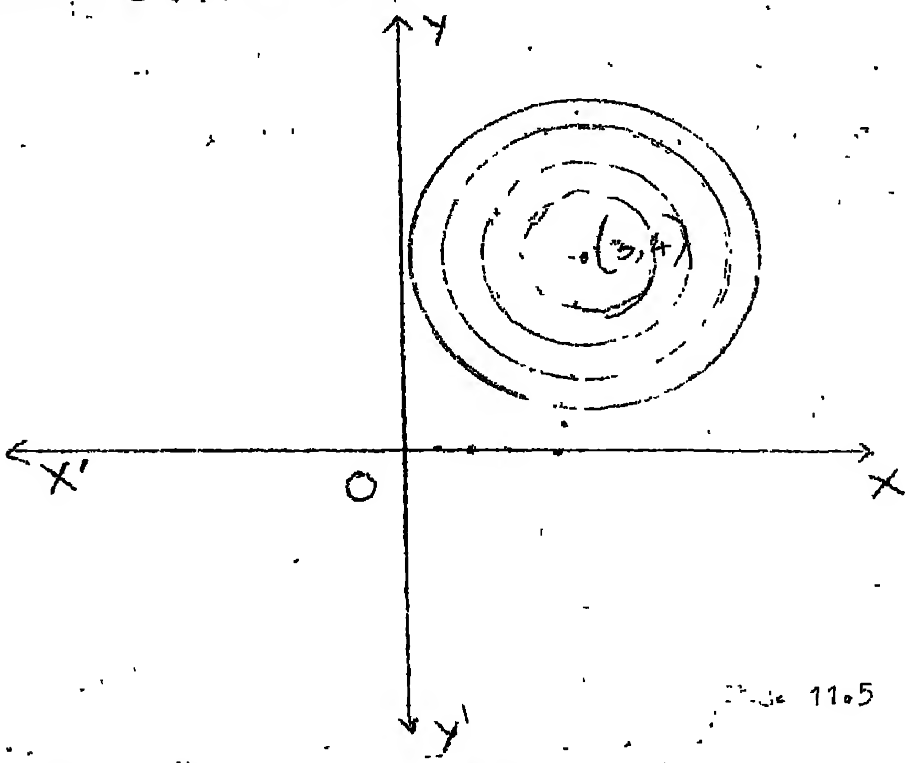


Fig. 11.5

It is interesting to note that through two given points sufficiently large number of circles can be drawn

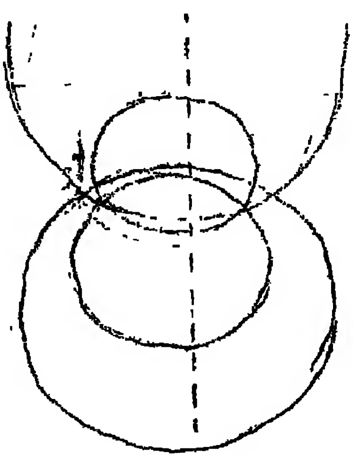


Fig. 11.6

We have already investigated in the last chapter that through the point of intersection of two lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$, sufficiently large number of lines can be drawn. The equation of this family of lines is given by

$A_1x + B_1y + C_1 + \lambda (A_2x + B_2y + C_2) = 0$ where λ is an arbitrary constant.

Consider

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + C_1 = 0$$

and $$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + C_2 = 0$$

be a pair of circles.

The equation

$$S_1 + \lambda S_2 = 0$$

or $$x^2 + y^2 + 2g_1x + 2f_1y + C_1 + \lambda (x^2 + y^2 + 2g_2x + 2f_2y + c_2) = 0$$

or $$x^2 + y^2 + 2x \frac{(g_1 + \lambda g_2)}{1+\lambda} + 2y \frac{(f_1 + \lambda f_2)}{1+\lambda} + \frac{C_1 + \lambda C_2}{1+\lambda} = 0$$

obviously represents a family of circles whose centre is

$\left(-\frac{g_1 + \lambda g_2}{1+\lambda}, -\frac{f_1 + \lambda f_2}{a+\lambda}\right)$ and

radius is $\frac{1}{1+\lambda}\sqrt{(g_1 + \lambda g_2)^2 + (f_1 + \lambda f_2)^2 - (1+\lambda)(c_1 + \lambda c_2)}$

$\forall \lambda \in R.$

11.8 Condition for Two Intersecting Circles to be Orthogonal

Let $S_1 = 0$ and $S_2 = 0$, be the equations of two intersecting circles, and PQ be the common chord. C_1 and C_2 are the centres, and r_1 and r_2 are the radii of the circles.

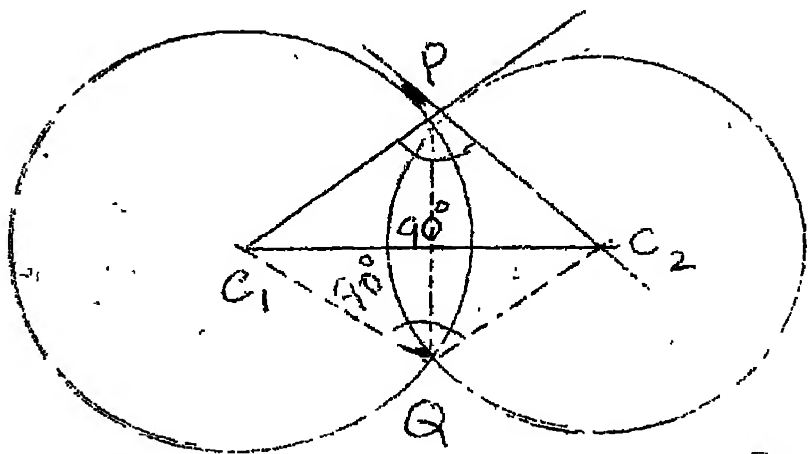


Fig. 11.7

These two intersecting circles will be orthogonal if the angles between the tangents of each circle at the point P and at the point Q are simultaneously at right angles. The knowledge of simple geometry will enable students to comprehend that PC_2 the radius of the second circle will be the tangent of the first circle and PC_1 the radius of the first circle will be the tangent of the second circle, and the angle $C_1PC_2 = 90°$, students who have learnt that $C_1\ (-g_1, -f_1)$, $C_2\ (-g_2, -f_2)$.

$$r_1 = \sqrt{(g_1^2 + f_1^2 - c_1)} \quad \text{and} \quad r_2 = \sqrt{g_2^2 + f_2^2 - c_2}.$$

Applying the Pythagoras theorem in the triangle PC_1C_2, we have

$$PC_1^2 + PC_2^2 = C_1C_2^2$$

or $$r_1^2 + r_2^2 = C_1C_2^2.$$

Here $C_1C_2^2 = (g_1 - g_2)^2 + (f_1 - f_2)^2$ substituting the values of r_1 and r_2, the distance formula gives the equation.

$$g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2$$

$$= (g_1 - g_2)^2 + (f_1 - f_2)^2$$

On simplification we obtain

$2\, g_1 g_2 + 2\, f_1 f_2 = c_1 + c_2$, which is the desired condition for intersecting circles $S_1 = 0$ and $S_2 = 0$ to be orthogonal. The circles $S_1 = 0$ and $S_2 = 0$ are said to intersect orthogonally in this case.

3. LEARNING OUTCOMES

(a) Essential Learning Outcomes for All : Comprehensive Study of this chapter enables each and every student of the class to be able to ,

(i) Write down the equation of a circle, when the centre and radius of the circle are given,

(ii) Find out the centre and radius of the circle, when its equation is given,

(iii) Comprehend that the general equation of second degree $ax^2 + 2\,hxy + by^2 + 2\,gx + 2\,fy + c = 0$ represents a circle if $h = 0$ and $a = b$. All students can find out the centre and the radius of the circle represented by the equation $ax^2 + ay^2 + 2gx + 2fy + c = 0$, correctly.

(iv) Obtain the equation of a circle passing through three given points.

(v) Write down the equation of the tangent of the circle, at a given point on the circle.

(vi) Calculate the length of the tangent from a given point on the given circle.

(vii) Investigate the tangency of a given line on a given circle. They can find the point of contact of a tangent to the circle.

(b) Learning Outcomes for the Higher Group : The students of higher group should be able to solve all the questions of application level confidently. They should be able to pick up the particular circle from a family of circles.

(4) TEACHING STRATEGIES

Motivation :. The previous knowledge of high school level may be successfully employed to motivate the students to study this chapter. Ask the students to draw circles of radii 2, 3, 4 and 5 units having centres say (3,4), (-3, -5), (-3, 5) and (3, -5): By taking a variable point (x, y) on each of these circles students should be prompted to write down the equation of each circle by using the distance formula. By expanding some of these equations, they should be motivated to observe the peculiarities of the equations such as absence of the term

containing xy, and equality of the co-efficients of x^2 and y^2. The students will be able to investigate the general equation of a circle.

$$x^2 + y^2 + 2\,gx + 2\,fy + c = 0.$$

Previous knowledge of completing the squares should be employed to put the general equation of a circle in the form

$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c.$$

This in turn will enable them to investigate the centre and radius of the circle represented by the general equation. Previous knowledge of synthetic geometry should be frequently employed to maintain the interest in the development of the lessons of this chapter.

Misconceptions / Common Errors.

(1) Generally most of the students fail to report the centre and radius of a circle, when its equation in the following forms are given

$$ax^2 + ay^2 + 2gx + 2fy + c = 0.$$

On dividing it by a, the above equation assumes the form

$$x^2 + y^2 + \frac{2g}{a}x + \frac{2f}{a}y + \frac{c}{a} = 0,$$

which in turn will enable to report the centre and radius of the circle correctly.

(2) The students should be asked to observe carefully that the centre of the circle ,

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is $(-g, -f)$. In exercises the equations of the circles in most of the cases are not given in accordance with the general form. The students should be asked to put the equation in suitable form so that mistakes may be avoided. For instance while reporting the centre of the circle $x^2 + y^2 - 10x + 15y + 6 = 0$, the student should be asked to put it as

$$x^2 + y^2 + 2(-5)x + 2\left(\frac{15}{2}\right)y + 6 = 0$$

Centre of this circle is $\left(5, -\frac{15}{2}\right)$ and not $\left(-5, \frac{15}{2}\right)$.

(3) While solving most of the problems of this chapter, putting the equation of the circle in proper standard form is a must to avoid general mistakes.

Solution/Hints for difficult Problems :

Exercise 11.1

Q.No. 3. Let $(x_1, 0)$ be the centre of the (.) required.

From question

radius $r = 5$ and the Circle passes through $(2, 3)$

$$(x_1 - 2)^2 + 9 = 25$$

or $(x_1 - 2)^2 = 16$

or $x_1 - 2 = \pm 4$

$x_1 = 6$ or, -2.

(6, 0) and (-2, 0) will be the centres of two separate circles having 5 as the length of its radius.

Equations of these circles are

$$(x - 6)^2 + y^2 = 25 \text{ and } (x + 2)^2 + y^2 = 25$$

or $$x^2 - 12x + y^2 + 11 = 0, \text{ and } x^2 + y^2 + 4x = 21.$$

Exercise 11.2

Q.No. 2 (a) Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the equation of the required circle. As it passes through origin and hence $c = 0$

For the points of intersection of the $\odot$ on the x axis put $y = 0$

$\therefore \quad x^2 + 2gx = 0$

or $\quad x(x + 2g) = 0$

Its intercept on the x-axis $= -2g$

$$\therefore \quad -2g = a$$

For the points of intersection of the circle with y-axis putting $x = 0$ we have

$$y^2 + 2fy = 0$$

$\therefore$ Intercept on the y-axis $= -2f$

$$\therefore \quad -2f = b$$

Hence equation of the required circle is

$$x^2 + y^2 - ax - by = 0$$

Exercise 11.3

Q.No. 1 Part-III

Equation of the given circle is

$$x^2 + y^2 + px + py = 0$$

or $$\left(x - \frac{p}{2}\right)^2 + \left(y - \frac{p}{2}\right)^2 = \frac{p^2}{2}$$

or $$\left(x - \frac{p}{2}\right)^2 + \left(y - \frac{p}{2}\right)^2 = \left(\frac{p}{\sqrt{2}}\right)^2$$

Hence equation of the circle in parametric form is

$$x = \frac{p}{2} + \frac{p}{\sqrt{2}} \cos\theta$$

$$y = \frac{p}{2} + \frac{p}{\sqrt{2}} \sin\theta .$$

Q.No. 2

(iv) Eliminating p between given two equations we have

$$2\,(2x - 2) = y + 1$$

or $y = 4x - 5$, which is a linear equation and hence represents a straight line.

Exercise 11.4

Q.No. 2 Tracing the lines $x = 6$, $x = -3$, $y = 3$ and $y = -1$, we have ,

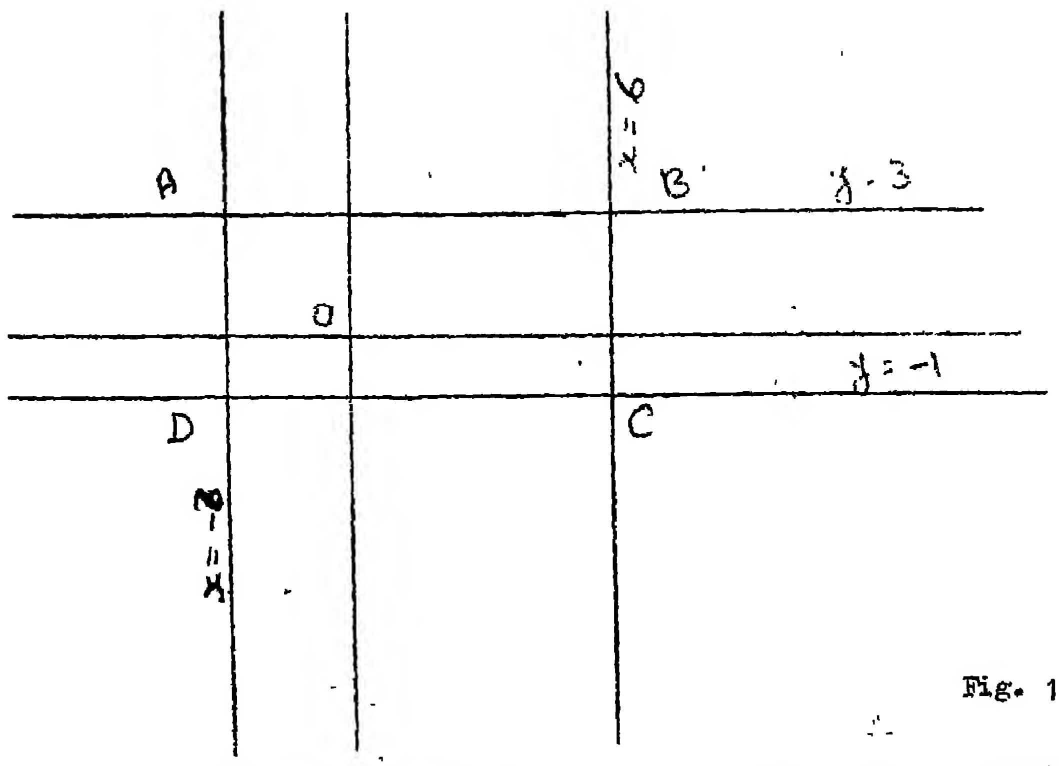


Fig. 11.8

The rectangle formed by the given four lines is ▭ A B C D.

Evidently $A \equiv (-3, 3)$, $B (6, 3)$

$C (6, -1)$, $D = (-3, -1)$

∴ Equation of the circle having AC as one of its diametres

is $(x + 3)(x - 6) + (y - 3)(y + 1) = 0$

or $x^2 - 3x - 18 + y^2 - 2y - 3 = 0$

or $x^2 + y^2 - 3x - 2y - 21 = 0.$

Similarly equation of the circle having BD as one of its diametre is $(x - 6)(x + 3) + (y - 3)(y + 1) = 0$

or $x^2 + y^2 - 3x - 2y - 21 = 0$

Exercise 11.8

Q.No.1 - Solution

Let the equation of the circle passing through (0, a) and (0, −a) be ,

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

$\therefore \quad 0 + a^2 + 0 + 2fa + c = 0$

or $\quad a^2 + 2fa + c = 0 \qquad \ldots\ldots\ldots (1)$

And $0 + a^2 + 0 - 2af + c = 0$

or $\quad a^2 - 2af + c = 0 \qquad \ldots\ldots\ldots (2)$

Solving (1) and (2) we have

$f = 0$ and $c = -a^2$

Thus $x^2 + y^2 + 2gx - a^2 = 0$ is the equation of family of circles passing through (0, a) and (0, −a). The condition for the tangency of $y = mx + c$ to the circle

$x^2 + y^2 + 2gx - a^2 = 0$ is

$$\frac{-mg + c}{\sqrt{1+m^2}} = \sqrt{g^2 + a^2}$$

or $\quad (c - mg)^2 = (g^2 + a^2)(1 + m^2)$; or, $c^2 + \cancel{m^2g^2} - 2mgc = g^2 + \cancel{m^2g^2} + a^2m^2 + a^2$

or $\quad g^2 + 2mgc + a^2m^2 + a^2 - c^2 = 0$

which is quadratic in g, consequently it will have two roots say g_1 and g_2.

$\therefore \quad g_1 + g_2 = -2mc \qquad \text{———} (3)$

$\quad g_1 g_2 = (a^2m^2 + a^2 - c^2) \qquad \text{———} (4)$

Hence there will be two circles

$$x^2 + y^2 + 2g_1 x - a^2 = 0$$

and $x^2 + y^2 + 2g_2 x - a^2 = 0$, which will pass through $(0, a)$ and $(0, -a)$ and touch the line $y = mx + c$.

These two circles will cut orthogonally if.

$$2g_1 g_2 + 2f_1 f_2 = c_1 + c_2$$

or $\quad 2 (a^2m^2 + a^2 - c^2) + 0 = -2 a^2$

or $\quad a^2m^2 + 2a^2 - c^2 = 0$

or $\quad c^2 = 2a^2 + a^2m^2$

or $\quad c^2 = a^2 (2 + m^2)$.

Question No.3

Let two circles $S = 0$ and $S^1 = 0$ be as per convention ,

$$x^2 + y^2 + 2 gx + 2 fy + c = 0 ,$$

and $\quad x^2 + y^2 + 2g^1 + 2f^1y + c^1 = 0$

$\therefore \quad r^2 = g^2 + f^2 - c$

$$g^2 + f^2 = r^2 + c \qquad \text{——— (1)}$$

$$g^{1^2} + f^{1^2} = r^{1^2} + c^1 \qquad \text{——— (2)}$$

Equations of two given circles are

$$\frac{S}{r} \pm \frac{S^1}{r^1} = 0$$

or $\quad r^1 S \pm r S^1 = 0$

or $\quad x^2 + y^2 + 2x \left(\dfrac{gr^1 \pm rg^1}{r^1 \pm r}\right) + 2y \left(\dfrac{fr^1 \pm rf^1}{r^1 \pm r}\right)$

$$+ \frac{cr^1 \pm cr^1}{r^1 \pm r} = 0$$

These circles will intersect orthogonally if

$$2\,\frac{g^2 r^{1^2} - r^2 g^{1^2}}{r^{1^2} - r^2} + 2\,\frac{(f^2 r^{1^2} - r^2 f^{1^2})}{r^{1^2} - r^2}$$

$$= \left(\frac{cr^1 + c^1 r}{r^1 + r}\right) + \frac{(cr^1 - c^1 r)}{r^1 - r}$$

or $$2r^{1^2}(f^2 + g^2) - 2r^2(f^{1^2} + g^{1^2}) = (cr^1 + c^1 r)(r^1 - r)$$

$$+ (cr^1 - c^1 r)(r^1 + r)$$

or $$2r^{1^2}(f^2 + g^2) - 2r^2(f^{1^2} + g^{1^2})$$

$$= r^{1^2}(c + c) - r^2(c^1 + c^1) + r^1 r(-c + c^1 + c - c^1)$$

or $$r^{1^2}(f^2 + g^2) - r^2(f^{1^2} + g^{1^2}) = cr^{1^2} - r^2 c^1$$

Substituting the values from (1) and (2) we have

$$r^{1^2}(r^2 + c) - r^2(r^{1^2} + c^1) = cr^{1^2} - r^2 c^1$$

or $$cr^{1^2} - c^1 r^2 = cr^{1^2} - r^2 c^1$$

which is true.

5. CHAPTER TEST

(a) Oral Test

Sample questions for oral test are given below. Readers are requested to set more questions of this type and use them while teaching the class.

(i) Find the equation of the circle whose centre is $(-3, 2)$ and radius is 3.

Ans: $(x + 3)^2 + (y - 2)^2 = 9$

or $x^2 + 6x + y^2 - 4y + 4 = 0$

(ii) Find the centre and radius of the following circle

$x^2 + y^2 - 4x + 6y + 3 = 0$

Ans: Centre $(2, -3)$.

radius $= \sqrt{10}$

(iii) Find equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point $(a \cos \alpha, a \sin \alpha)$

Ans : $x \cos \alpha + y \sin \alpha = a$.

b) Written Test

Two separate questions are given below: Teachers are requested to set questions to meet the needs.

(1) ON the line joining $(1, 0)$ and $(3, 0)$ as base, an equilateral Δ is formed having its vertex in the positive quadrant. Find the equations to the circles described on it as diameter.

Ans: $x^2 + y^2 - 4x + 3 = 0$. $x^2 + y^2 - 3x - \sqrt{3}\, y + 2 = 0$

$x^2 + y^2 - 5x - \sqrt{3}\, y + 6 = 0$.

(2) If $y = mx$ be the equation of the chord of a circle whose radius is a, the origin being one extremity of the chord and Axis of X being a diametre of the circle. Prove that the equation of the circle of which this chord is a diametre is

$(1 + m^2)\ (x^2 + y^2) = 2a\ (x + my\ )$

References: As given in the chapter 8 on cartesian - plane.

CHAPTER 12

CONIC SECTION

1. Introduction

In the preceding chapter, an exposition has been made regarding conic-sections or conics. Four types of curves, the circle, the ellipse, the parabola, and the hyperbala are categorised as conics or conic-sections as these are obtained by intersecting a double right circular cone with a plane inclined at different angles to the axis of the cone. If included angle between the intersecting plane and the axis of the cone is equal to the semi-vertical angle of the cone, the conic obtained is a parabola otherwise the conic is an ellipse or a hyperbola depending on whether the plane cuts just one or both nappes of the double-cone. The hyperbala is to be thought of just a single curve consisting of two branches, one on each nappe. If the plane of Intersection is at right angle to the axis of the cone, the conic thus obtained is a circle, which has been discussed in Chapter 11. In this chapter common properties of Parabola, ellipse and hyperbola are investigated.

2. Content Analysis

In this section the number of each subsection is in accordance with the Text Book.

12.1 It is interesting to note that if the intersecting plane is moved parallel to itself, so that it passes through the vertex of the cone we get degenerate conics. In the case of the parabola the

the degenerate conic is a line and in the case of ellipse the degenerate conic will be a point while in the case of hyperbola the degenerate conic will be a pair of lines.

The ellipse, the hyperbola and the parabola will be studied here as plane figures and the characteristics of these curves will be investigated with respect to two fixed points called foci in the case of ellipse and hyperbola, and a fixed point called the focus and a fixed line in the case of the curve parabola.

The ellipse is defined to be a locus of such a point, whose distance from two points F_1 and F_2 say d_1 and d_2 are such that $d_1 + d_2 = 2a$ a constant. Two fixed points F_1 and F_2 are called the foci of the ellipse.

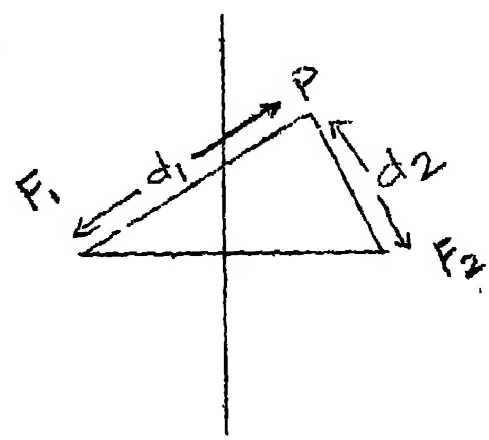
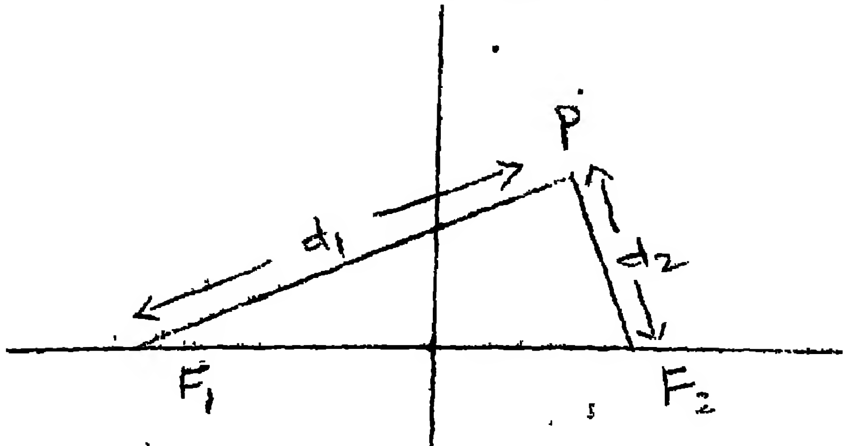


Fig. 12.1

Let F_1 and F_2 be the foci of the ellipse and P be a point on it. Hence from definition of an ellipse $PF_1 + PF_2 = 2a$, a constant.

The hyperbola is the locus of such a point, so that the difference of its distances d_1 and d_2 from two points F_1 and F_2 is a constant say 2a. Analogous to ellipse the two fixed points F_1 and F_2 are called the foci of the hyperbala.



$|d_1 - d_2| = 2a$ (a constant)

Fig. 12.2

The parabala is the locus of such a point whose distances d_1 and d_2 from a fixed point F , and a fixed line MN are equal. The fixed point F is called the focus of the parabala and MN is called the directrix of the parabala.

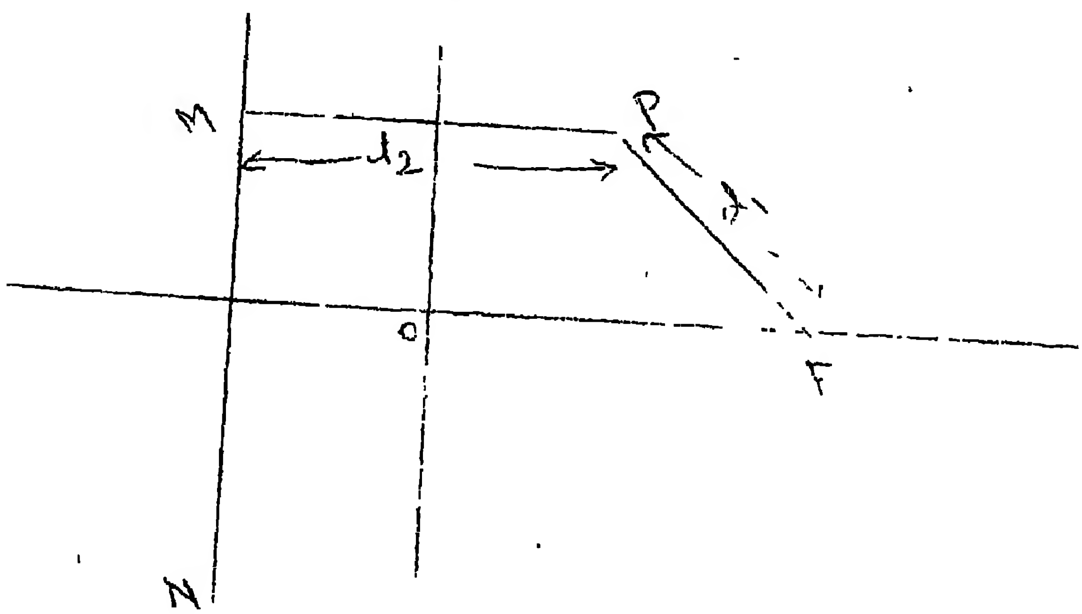


Fig. 12.3

12.2 Parabala

Definition

A parabola is the locus of a point whose distances from a fixed point and from a fixed line are equal.

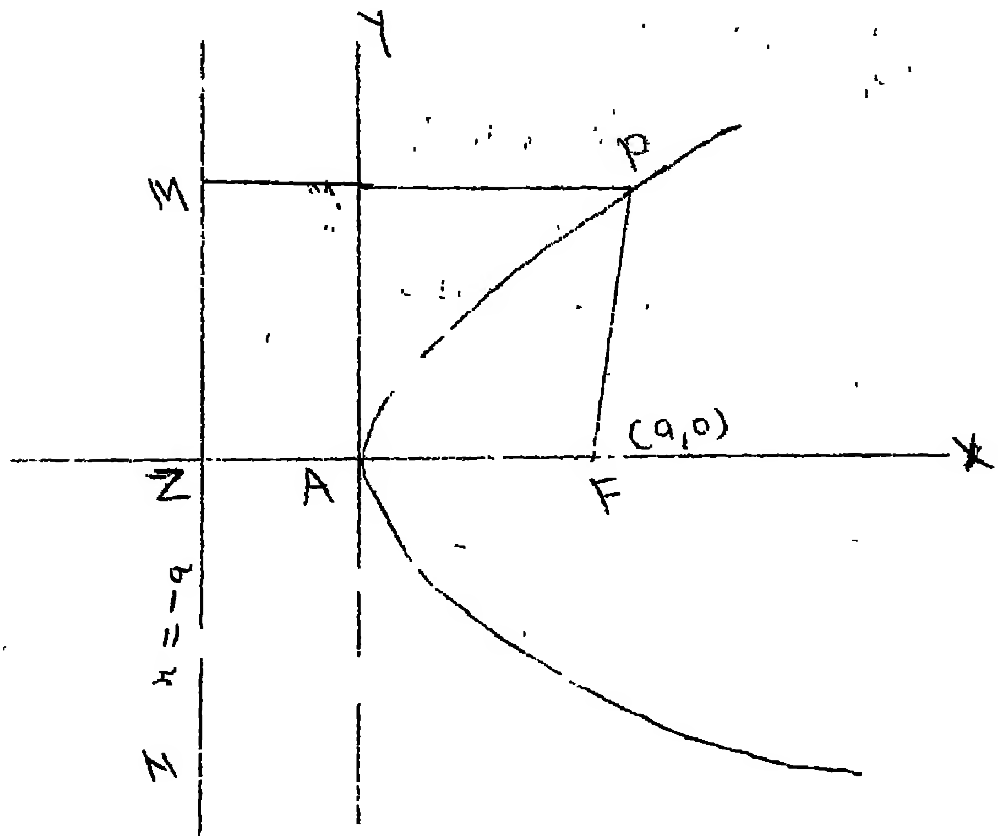


Fig. 12.4

Let F be the focus and MN the Directrix of the parabola. FZ is the perpendicular on MN. As F is fixed and MN is also fixed and hence FZ is also fixed. The line FZ, which is fixed, is known as the axis of the parabola.

The measure of the linessegment is also fixed. Let FZ = 2a. A the middle point of FZ is also fixed. As the distance of A from F is equal to the distance of MN from A, consequently A lies on the parabola. Thus A is the only point of the parabola, which lies on AZ, the axis of the parabola. The point A is called the vertex of the parabola.

Taking A as origin, and AY perpendicular to AZ as the axis of Y we find F (a, 0) and Z = (−a, 0).

Equation of MN the directrix is $x = -a$.

m

Let P (x, y) be any point on the parabola and hence from the definition of parabola.

PF = PM, where PM is perpendicular to MN.

Thus $PF^2 = PM^2$

or, $(y - 0)^2 + (x - a)^2 = (x + a)^2$

$\Longrightarrow \quad y^2 = 4ax$, which is the standard equation of parabola in simple and elegant form.

Replacing y by −y in the equation $y^2 = 4ax$, we find that it remains unaltered and hence we say that the parabola $y^2 = 4ax$ is symmetrical about x – Axis, the axis of the parabola.

Due care must be taken to prompt the students to study the parabola, while its equations are in the following forms :

(i) $x^2 = 4ay$

Vertex A $\equiv$ (0, 0)

Focus F $\equiv$ (0, a)

Equation of the Directrix of the parabola is

$y = -a$

Equation of the axis of the parabola is

$x = 0$ namely y - Axis.

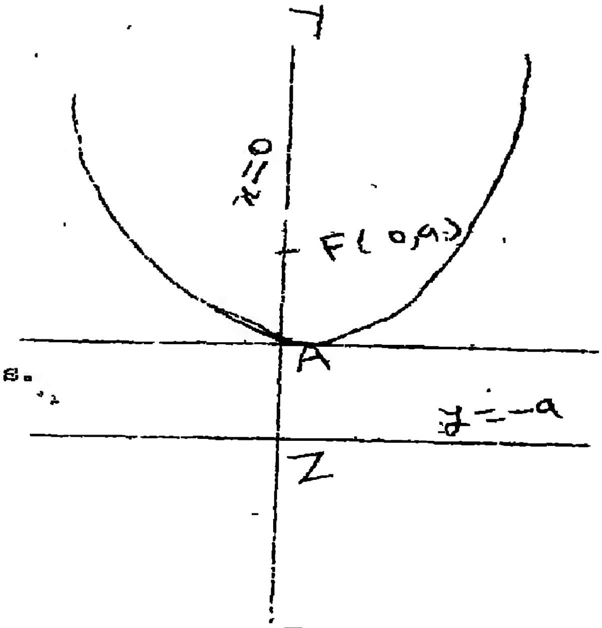


Fig. 12.5

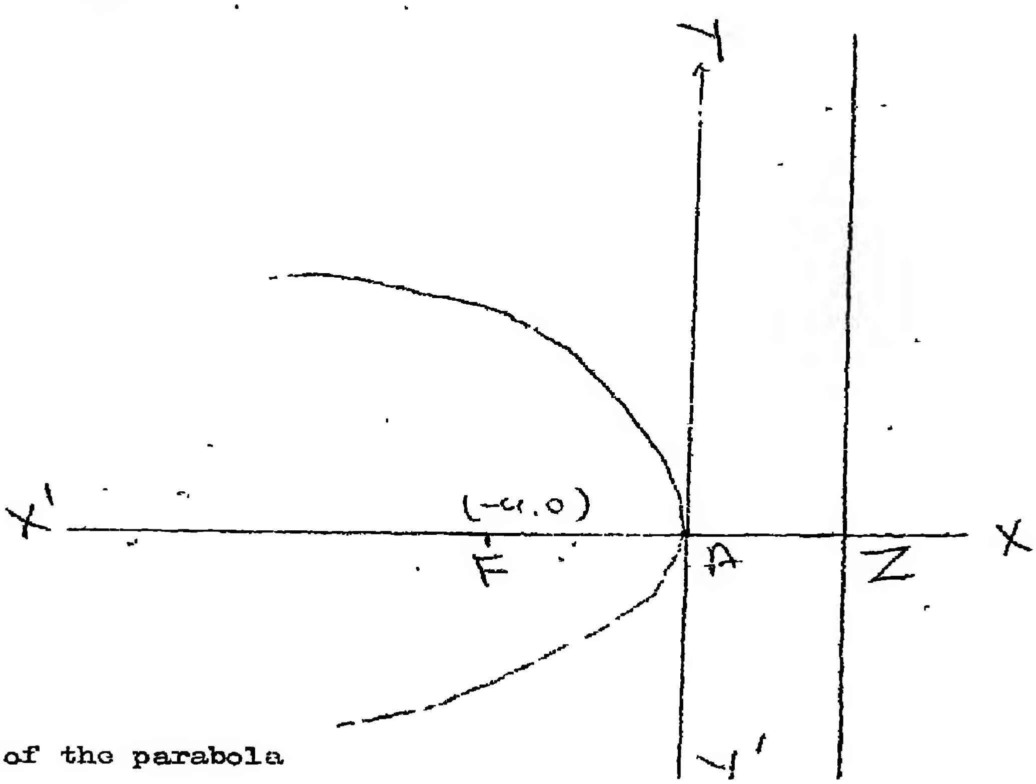
(ii) $y^2 = -4ax$

Vertex A $\equiv$ (0, 0)

Focus $\equiv$ (-a, 0)

Equation of the directrix of the parabola is

$x = a$



Axis of the parabola

(iii) $x^2 = -4aY$

Vertex A $\equiv$ (0, 0)

Focus F $\equiv$ (0, -a)

Equation to the corresponding directrix of the parabola is $y = a$.

Axis of the parabola is $x = 0$ namely y - Axis.

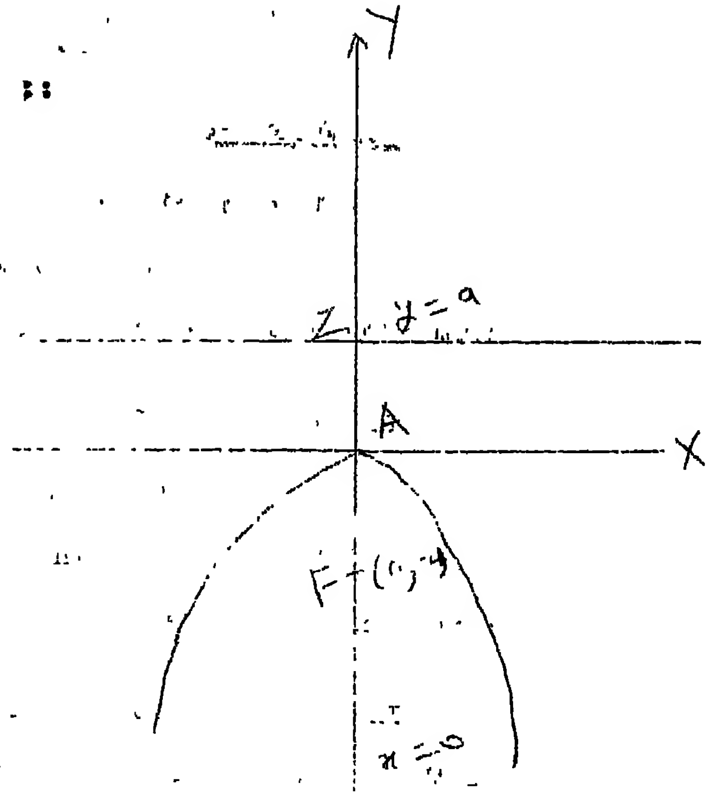


Fig. 12.7

Students should be prompted to note that the parabolas represented by the equations $y^2 = 4ax$, $x^2 = 4ay$, $y^2 = -4ax$ and $x^2 = -4ay$ lie in any two of the four quadrants of the cartesian-plane.

12.3 Ellipse

Definition

(i) An ellipse is the locus of a point, the sum of whose distances from two fixed points is a constant. The two fixed points are the foci of the ellipse.

Under the above definition of the ellipse the standard form of the equation of the ellipse is obtained in the Text Book.

Here an alternative method of defining the ellipse is given below :

Definition -2

An ellipse is the locus of a point whose distance d_1, from a fixed point F_1 bears a constant-ratio e, to its distance d_2 from a fixed line, where $e \angle 1$.

The fixed point F_1 is called the focus of the ellipse.

e the fixed ratio is called the eccentricity of the ellipse where $e \angle 1$. The fixed line is called the directrix of the ellipse corresponding to focus F_1 of the ellipse.

In view of the above definition of an ellipse standard form of the equation of the ellipse is given below :

Let F_1 be the focus and MN be the corresponding directrix of the ellipse. Draw F_1Z perpendicular on MN. Evidently A a point on F_1Z will divide it internally such that

$$AF_1 = e. \ AZ \quad \text{———} \quad (1)$$

Similarly the external division of F_1Z will give a point A^1 on F_1Z such that $F_1A^1 = e. \ A^1Z$ ——— (2)

As F_1 is the fixed point and MN is the fixed line and hence F_1Z is fixed. The external and internal division of F_1Z have given us two fixed points A^1 and A in the ratio of e : 1. Let $AA^1 = 2a$, and C be the middle point of AA^1.

$$AC = CA^1 = a .$$

Taking C as origin AA^1 as the x-axis and CY as the axis of Y, obtain

$$C \equiv (0, 0) \qquad A^1 \equiv (a, 0) \qquad A = (-a, 0)$$

Expressing the distances in terms of the distances measured from C in the equations (1) and (2) we have —

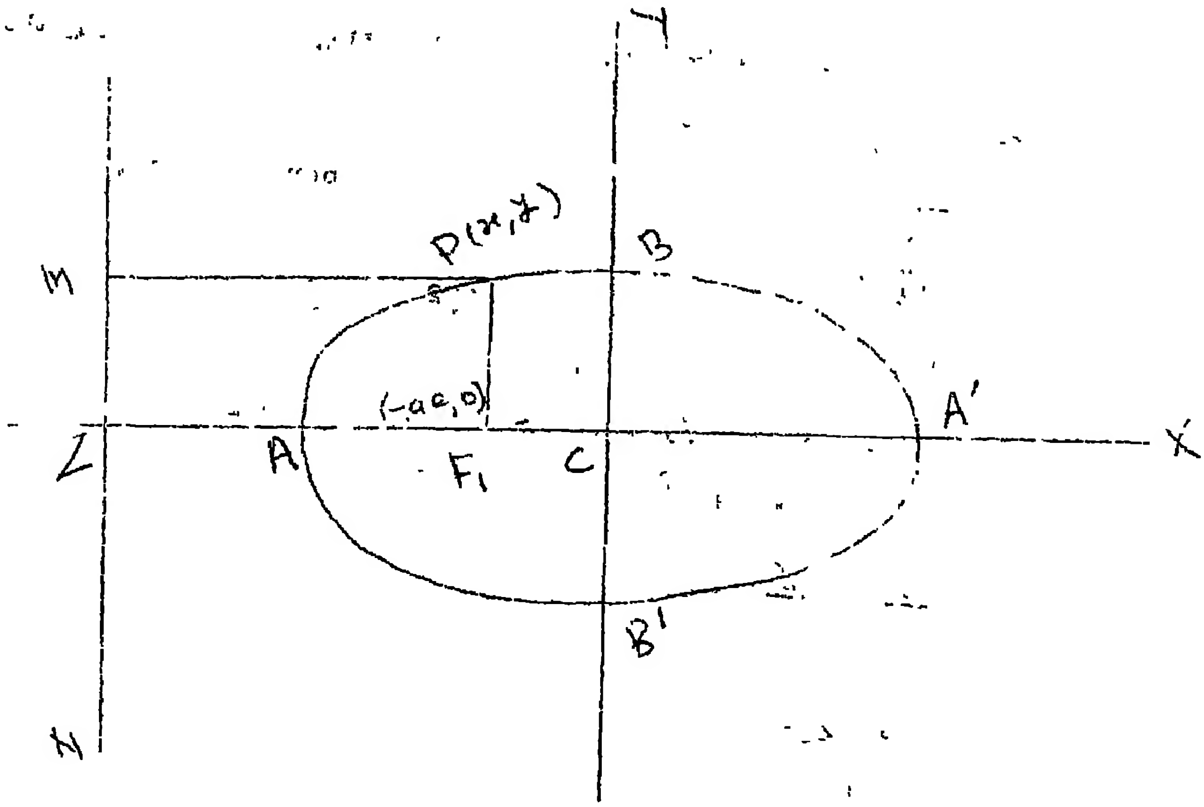


Fig. 12.8

$$CA - CF_1 = e\,(CZ - CA)$$

or $$a - CF_1 = e\,(CZ - a) \qquad \text{——— (1)}$$

And $$CF_1 + CA^1 = e\,(A^1C + CZ)$$

or $$CF_1 + a = e\,(a + CZ) \qquad \text{——— (2)}$$

Adding (1) and (2) we have

$$2a = 2e.\,CZ$$

$$\therefore\ CZ = \frac{a}{e} \qquad \text{———} \qquad (3)$$

Subtracting (1) from (2) we have

$$2\,CF_1 = 2ae$$

or $$CF_1 = ae \qquad \text{——— (4)}$$

Now we are in a position to write down the coordinates of F_1 and Z. Hence $F_1 \equiv (-ae, 0)$ and $Z \equiv (-\frac{a}{e}, 0)$.

Taking any point P (x, y) on the ellipse and applying the second definition, we have —

$PF_1 = e.\ PM$ where M is the foot of perpendicular on MN from P

or $PF_1^2 = e^2.\ PM^2$

$$(x + ae)^2 + y^2 = e^2 \left[x + \frac{a}{e}\right]^2$$

or $x^2 + a^2e^2 + 2aex + y^2 = e^2x^2 + a^2 + 2aex$

or $x^2 (1 - e^2) + y^2 = a^2 (1 - e^2)$

or $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 (1-e^2)} = 1.$

As $e < 1$ and $e^2 < 1$

$1 - e^2 > 0$

$a^2 (1-e^2) > 0$ let it be given by b^2.

$\therefore$ Standard equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{......} \ (A)$$

Replacing x and y respectively by −x and −y we find that the equation remains unaltered. This proves that the ellipse (A) is symmetrical about both the Axes.

The standard equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{———} \qquad (A)$$

we have seen that

$$b^2 = a^2 (1-e^2) = a^2 - a^2 e^2$$

$\therefore \quad b < a$

The ellipse (a) meets the axis of x at the point A and A^1 given by

$$\frac{x^2}{a^2} + 0 = 1$$

or $\quad x = \pm a$

$\therefore \quad A \equiv (-a, 0)$ and $A^1 \equiv (a, 0)$

Again the ellipse (A) meets the axis of y at the points B and B^1 given by

$$\frac{y^2}{b^2} = 1$$

$\quad y = \pm b$

$\therefore \quad B \equiv (0, b)$ and $B^1 \equiv (0, -b)$

$BB^1 = 2b$ and $AA^1 = 2a$

$AA^1 = 2a$ is called the major axis of the ellipse and $BB^1 = 2b$ is known as minor axis of the ellipse.

<u>Note:</u> Consider the equation

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

comparing it with (A) we have

$$a^2 = 9 \text{ and } b^2 = 16$$

Here $b^2 > a^2$

Therefore major axis of the ellipse will be along the axis of y. For finding ehe accentricity 'e' we have

$$9 = 16\ (1 - e^2)$$

or $\quad \frac{9}{16} = 1 - e^2$

or $\quad e^2 = \frac{7}{16}$

$$e = \frac{\sqrt{7}}{4}$$

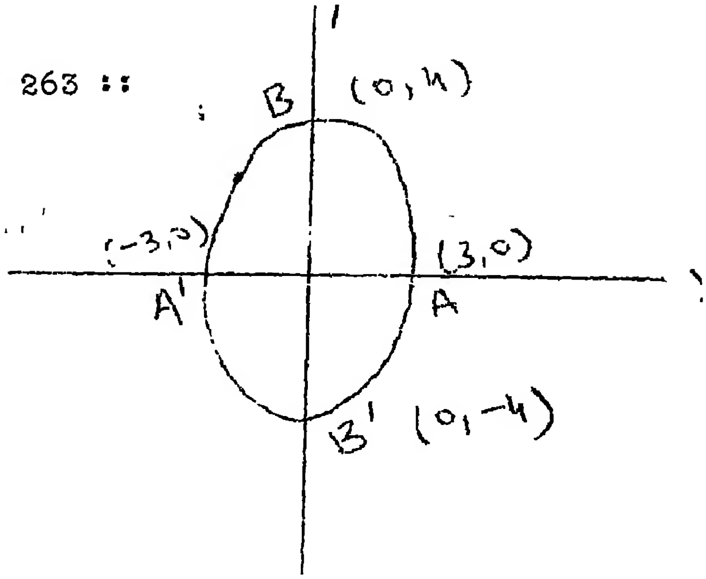


Fig. 12.9

From symmetry it is obvious that the ellipse has two foci F_1 and F_2 and two directrices corresponding to each focus.

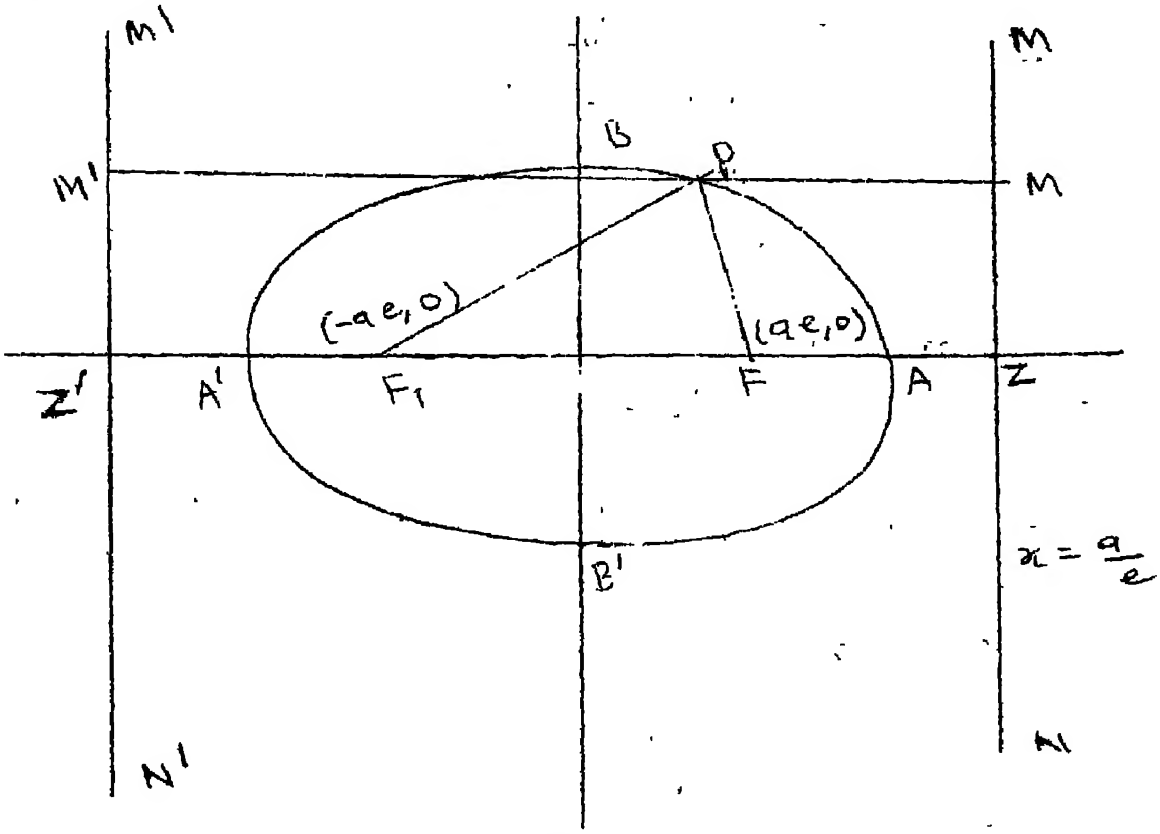


Fig. 12.10

corresponding to Focus F_1 $(-ae, 0)$ the directrix is $x = -a/e$ and $x = \frac{a}{e}$ is the equation of the directrix relative to the focus $F(ae, 0)$.

It is interesting to note that the sum of the focal distances of any of the ellipse is equal to 2a. Its proof is given below :

From the definition of the ellipse

$$PF_1 = e\left[PM'\right] \quad \text{——— (1)}$$

$$PF = e\left[PM\right] \quad \text{——— (2)}$$

Adding (1) and (2) we have,

$$PF_1 + PF = e\left[PM^1 + PM\right]$$

$$= e\left[M^1 M\right]$$

$$= e\left[Z^1 Z\right]$$

$$= e\left[\frac{2a}{e}\right] = 2a.$$

This result is in accordance with the first definition of the ellipse given in the text-book.

12.4 Hyperbala

As in the case of ellipse, two definitions of hyperbola are stated here.

Definition 1: Hyperbola is the locus of such a point, the difference of whose distances from two fixed points (foci) is a constant '2a'.

2. Hyperbola is the locus of such points, whose distances from a fixed point bear a fixed ratio 'e' with its distances from a fixed line where $e > 1$.

The fixed point say F_1 is called the focus, and the fixed line is called the directrix of the hyperbola, whereas the fixed ratio 'e' is known as the eccentricity of the hyperbola.

Following the first definition the standard Equation of Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is obtained in the textbook. Here we will obtain the same, keeping in view the second definition as previously done in the case of ellipse. Let F_1 be the focus and M_1N_1 be the corresponding directrix of the hyperbola. Draw F_1Z_1 perpendicular to M_1N_1, the directrix

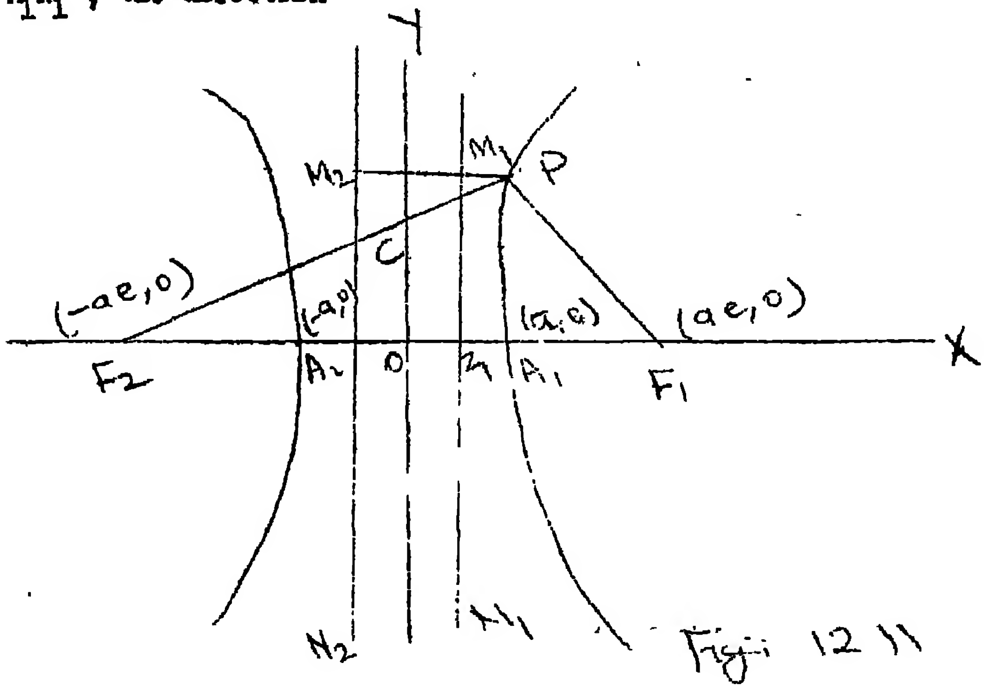


Fig. 12.11

Evidently F_1Z_1 is a fixed line, known as the axis of hyperbola. On F_1Z_1 a point A_1 will be obtained such that

$$F_1A_1 = e.\ A_1Z_1 \qquad \text{—— (1)}$$

As $e > 1$, it is interesting to note that $A_1Z_1 < F_1A_1$, where A_1 is an internal point of F_1Z_1. Evidently A_1 will lie on the hyperbola. From external division of F_1Z_1, a point A_2 will be obtained such that

$$F_1A_2 = e. \; A_2Z_1 \qquad \text{——— (2)}$$

Here A_2 will be on the left-hand side of A_1, unlike the ellipse where A_2 was on the right-hand side of A_1.

Evidently A_1 & A_2 are two fixed points of the axis of hyperbola which lie on it. Taking $A_1A_2 = 2a$, find the middle point O of A_1A_2

$$\therefore \; A_1O = OA_2 = a.$$

Taking O as origin and A_1A_2 as the axis of X, we have

$A_1 = (a, 0)$ and $A_2 = (-a, 0)$

Expressing (1) and (2) in terms of distances measured from O we have

$$OF_1 - OA_1 = e \; (OA_1 - OZ_1)$$

or $\quad OF_1 - a = e \; (a - OZ_1) \qquad$ ——— (1)

And $\quad OA_2 + F_1O = e \; (OA_2 + OZ_1)$

or $\quad a + F_1O = e \; (a + OZ_1) \qquad$ ——— (2)

Adding (1) and (2) we have —

$$OF_1 = ae \qquad \text{——— (3)}$$

substracting (2) from (1) we have

$$a = e. \; OZ_1$$

$$OZ_1 = \frac{a}{e} \qquad \text{——— (4)}$$

$$\therefore \; F_1 = (ae, 0) \quad Z_1 = \left(\frac{a}{e}, 0\right)$$

and hence equation to the directrix

M_1N_1 is $x = a/e$

Considering $P\ (x, y)$ any point on the hyperbola and applying the definition – 2, we have —

$$PF_1 = e.\ PM_1$$

or $$PF_1^2 = e^2.\ PM_1^2$$

or $$(x - ae)^2 + y^2 = e^2 \left[-x + \frac{a}{e}\right]^2$$

or $$x^2 + a^2e^2 - \cancel{2aex} + y^2 = e^2x^2 - \cancel{2aex} + a^2$$

or $$x^2 (1 - e^2) + y^2 = a^2 (1 - e^2)$$

or $$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1 \qquad \text{——— (A)}$$

As $e > 1$ and $e^2 > 1$ & hence $e^2 - 1 > 0$

It means $a^2 (1-e^2) < 0$

In order to make $a^2 (1-e^2)$ positive we should express $a^2 (1-e^2) = -a^2 (e^2-1)$. Hence equation (A) assumes the form

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$$

keeping $a^2 (e^2-1) = b^2$, it reduces to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, which is the standard equation of the hyperbola in simple and elegant form. Obviously its form is the same as that obtained in the textbook.

A_1A_2 is called the transverse axis of the hyperbola. The perpendicular bisector of A_1A_2 is called the conjugate axis of the hyperbola.

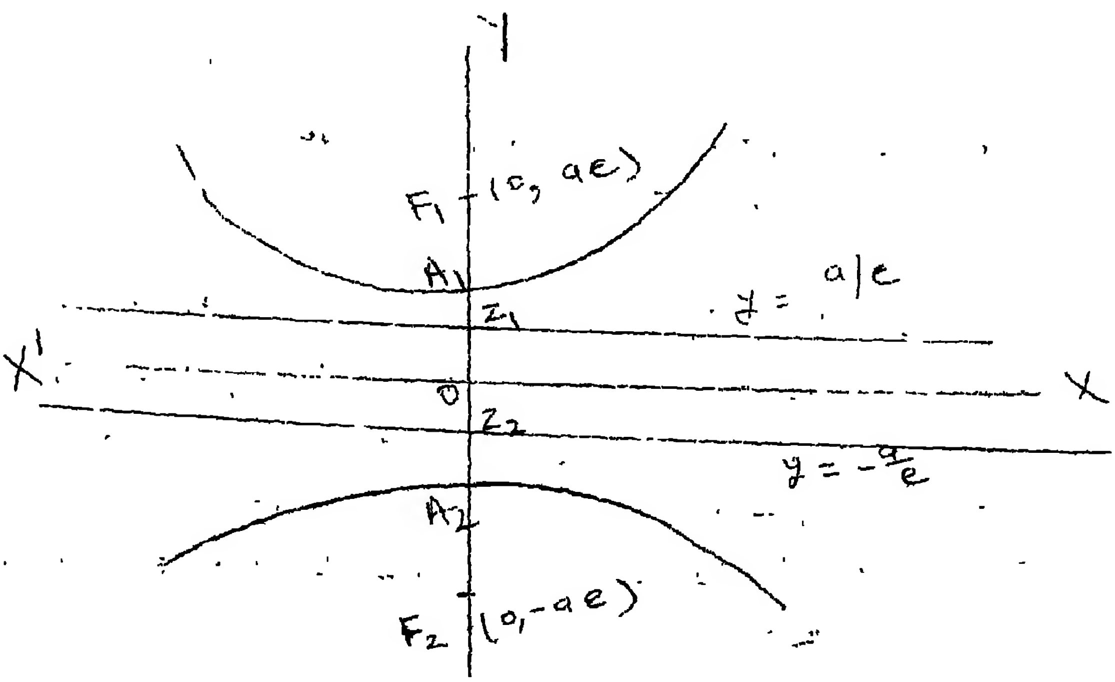
The points A_2 and A_1 of the transverse axis of the hyperbola are called vertices of the hyperbola. 2a, the distance between A_2A_1 is called the length of transverse axis of the hyperbola.

Replacing x & y by -x & -y in the equation

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we find that it remains unaltered, it means hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is symmetrical about both the axis of X and Y. It is interesting to note that hyperbola does not meet the conjugate axis, as the equation obtained by putting x = 0 is

$\frac{y^2}{b^2} = -1$, which does not have real roots.

If the foci F_1 and F_2 lie on y-axis, the shape of the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ will be as given below: (0, ae) and (0, -ae) will be the co-ordinates of F_1 and F_2 and $y = \frac{a}{e}$ and $y = -\frac{a}{e}$ will be corresponding directrices. The transverse axis will be along the axis of Y.



It is interesting to note that foci of the hyperbola represented by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, are along the axis of X whereas the foci of the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ will be along the axis of Y. Foci lie on the transverse axis only

Again $b^2 = a^2 (e^2-1)$ where $e > 1$

taking $e = \sqrt{2}$ we have

$$b^2 = a^2 (2-1)$$

or $b^2 = a^2$

It means $b = a$, evidently

$$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1 \quad \text{or,} \quad x^2 - y^2 = a^2$$

will also represent a hyperbola having its foci along x - axis, where the hyperbola represented by $y^2 - x^2 = a^2$ will have its foci along the axis of Y.

Hyperbola represented by $x^2 - y^2 = a^2$ and $y^2 - x^2 = a^2$ are called Rectangular Hyperbolas. The eccentricity of rectangular hyperbolas is always $\sqrt{2}$.

It is interesting to note that like ellipse, hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has two foci F_1 (ae, 0) and F_2 (- ae, 0) and corresponding to them two directrices. The equations of the directrices are respectively $x = \frac{a}{e}$ and $x = -\frac{a}{e}$. In order to establish the analogy in both the definitions of hyperbola, the following property is to be investigated.

The difference of the focal distances of any point on the hyperbola is constant.

Consider the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and let a point P be on any one of its branches. From P perpendiculars are drawn on the directrices M_1N_1 and M_2N_2. From definition of hyperbola (see Fig. 12.11)

$$PF_1 = e.\ PM_1$$

And $$PF_2 = e\ PM_2$$

$$\therefore \quad PF_2 - PF_1 = e\left[PM_2 - PM_1\right]$$
$$= e\left[M_2\ M_1\right]$$
$$= e.\ \frac{2a}{e}$$
$$= 2a.$$

Hence we find that two different definitions of the hyperbola lead to the same results, and are merely two ways of expressing the same characteristic.

12.5 Condition for tangency of the line $y = mx + c$

We have proved that the condition for the tangency of the line $y = mx + c$ to the circle $x^2 + y^2 = r^2$ is obtained by finding the condition for two coincident points of intersection of the circle $x^2 + y^2 = r^2$ and the line $y = mx + c$. Similar treatment will be adopted here to investigate the condition for tangency.

The line $y = mx + c$ ——— (1)

will be a tangent to the

parabola $y^2 = 4ax$ ——— (2).

if the points of intersections of (1) and (2) are real and coincident.

Solving (1) and (2) as simultaneous equation we have

$$(mx + c)^2 = 4ax$$

or $$m^2x^2 + 2x(mc - 2a) + c^2 = 0$$

Both the roots of this equation will be real and coincident if

$$(mc - 2a)^2 = m^2 c^2$$

or $$4a^2 - 4amc = 0$$

or $$a - mc = 0$$

or $$c = a/m$$

Thus we find that $y = mx + a/m$ is the equation of any tangent of the parabola $y^2 = 4ax$. The point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

It may be noted that $x = at^2$ and $y = 2at$ is the parametric equation of the parabola $y^2 = 4ax$. Similar treatment will enable us to find the condition for tangency of the line $y = mx + c$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The required condition is

$$c = \pm\sqrt{a^2 m^2 + b^2}$$

Thus $y = mx \pm \sqrt{a^2 m^2 + b^2}$ is the equation of any tangent of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The point of the contact of the tangent $y = mx + \sqrt{a^2 m^2 + b^2}$ is

$$\left[\frac{-a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{b^2}{\sqrt{a^2m^2 + b^2}}\right]$$

It is worthful to observe that $(a \cos \phi, b \sin \phi)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $x = a \cos \phi$ and $y = b \sin \phi$ is the parametric equation of the ellipse.

Similarly the condition for tangency of the line $y = mx + c$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $c = \pm \sqrt{a^2m^2 - b^2}$

Thus $y = mx + \sqrt{a^2m^2 - b^2}$ is the equation to any tangent of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, having $\left(\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \frac{b}{\sqrt{a^2m^2-b^2}} \right)$ as it point of contact.

The parametric equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x = a \sec \phi$ and $y = b \tan \phi$, which can easily be verified.

3. Learning Outcomes

(a) Essential Learning Outcomes for all.

Comprehensive study of the content of this chapter will enable the students.

(i) To spell out how the conic sections are formed by the intersection of a plane with a double right circular cone and identify the conic.

(2) To draw a rough sketch of all sorts of conics, locating axes, foci, directrices, centre, while the equations of the conics are given in standard forms.

(3) To know that eccentricity of the parabola is $e = 1$. In the case of ellipse eccentricity $e < 1$, and $e > 1$ while the conic is hyperbola.

(4) To know that the equation of the ellipse in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given and $a^2 > b^2$, then the foci of the ellipse will lie on x-axis, while foci will lie on y-axis if $b^2 > a^2$.

(5) To know that the transverse axis of the hyperbola represented by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, will be along x-axis, while the transverse axis of the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ lies on y-axis. Foci of the hyperbola lie on the transverse axis only.

(6) To know that the equations $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{a^2} = 1$ represent rectangular hyperbolas. The eccentricity of the rectangular hyperbola is $\sqrt{2}$.

(b) <u>Learning Outcomes for the higher group.</u>

The students of the higher group will be able to find the condition for the tangency of a given line to a given conic. If the condition for tangency is satisfied they will be able to find the point of contact.

4. <u>Teaching Strategies</u>

<u>Motivation.</u> The previous knowledge gained by now may be employed to motivate the students to study the current chapter. They have learnt that the general equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{—— (A)}$$

represents a pair of lines if

$\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$. They too have learnt that (A) represents a circle if $\Delta \neq 0$ and $h = 0$ and $a = b$. In other words the equation $ax^2 + ay^2 + 2gx + 2fy + c = 0$

$$\Longrightarrow x^2 + y^2 + \frac{2g}{a}x + \frac{2fy}{aa} + \frac{c}{a} = 0$$

represents a circle. Further the general equation of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$

$$\Longrightarrow (x+g)^2 + (y+f)^2 = g^2 + f^2 - c$$

On translating the axes the equation of the above form are transformed into :-

$$X^2 + Y^2 = R^2.$$

The parametric form of this equation is

$$X = R\cos\varphi \text{ and}$$
$$Y = R\sin\varphi.$$

Now an intense longing among the students can be aroused to study the curve whose parametric equation is $x = a\cos\theta$ and $y = b\sin\theta$. Taking $a = 10$ and $b = 6$, a table of the following type is to be developed.

φ	x	y	(x, y)
0°	10	0	$(10, 0)$
30°	$\frac{10\sqrt{3}}{2}$	$6 \times \frac{1}{2}$	$(5\sqrt{3}, 3)$
45°	$\frac{10}{\sqrt{2}}$	$\frac{6}{\sqrt{2}}$	$(5\sqrt{2}, 3\sqrt{2})$
60°	5	$3\sqrt{3}$	$(5, 3\sqrt{3})$
90°	0	6	$(0, 6)$
120°	–	–	–
135°	–	–	–
180°	–	–	–

Taking sufficiently large number of points and plotting them on a cartesian plane, the shape of ellipse can be obtained. It will prompt the students to study the ellipse and other conics as well.

Misconceptions/common Errors

(i) While tracing the parabolas, when its equations in any one of the four standard forms $Y^2 = \pm 4aX$ or $X^2 = \pm 4aY$ are given, and reporting the focus, the vertex and the directrix a comparative study of the parabolas is a must.

(2) In order to transform a given equation in to the standard forms the knowledge of translation of the axes is essential. Sufficient practice and training leads to success.

(3) While reporting the axes of the ellipse say $\frac{x^2}{36} + \frac{y^2}{64} = 1$, it is obvious that $64 > 36$ and hence major axis of the ellipse $\frac{x^2}{36} + \frac{y^2}{64} = 1$ will be along Y-axis, while the major axis of the ellipse $\frac{x^2}{64} + \frac{y^2}{36} = 1$ will be along x axis. The shapes of the ellipses in both the cases shapes and co-ordinates of foci, major and minor axes, eccentricity, and vertices should be clarified with the help of sufficient amount of examples and practices. Comparative Study of both the ellipses is worthful.

(4) While reporting the foci of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ it must be noted that foci will lie along x-axis. Whereas, the foci of the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ will lie along y-axis. Comparative study of the hyperbolas represented by two equations above must be done to clarify the concept in order to eradicate the mistakes.

Solution/hints to difficult problems.

Exercise 12.1

Q. 1(c) Given equation is

$$y^2 = -12x$$

$$\text{or} \quad y^2 = -4\,(3)x$$

Focus $\equiv (-3, 0)$ and equation to the directrix is $x = 3$

Q. 1(d) Given equation is

$$x^2 = -16y$$

$$\text{or} \quad x^2 = -4\,(4)y$$

Focus $\equiv (0, -4)$ and equation to the Directrix is $y = 4$.

Q. 2(b) Directrix is $y = 2$

and vertex is $(0, 0)$

$\therefore$ Focus $F = (0, -2)$.

The equation of the parabola is

$$x^2 = -4\,(2)\,y$$

or $$x^2 = -8\,y.$$

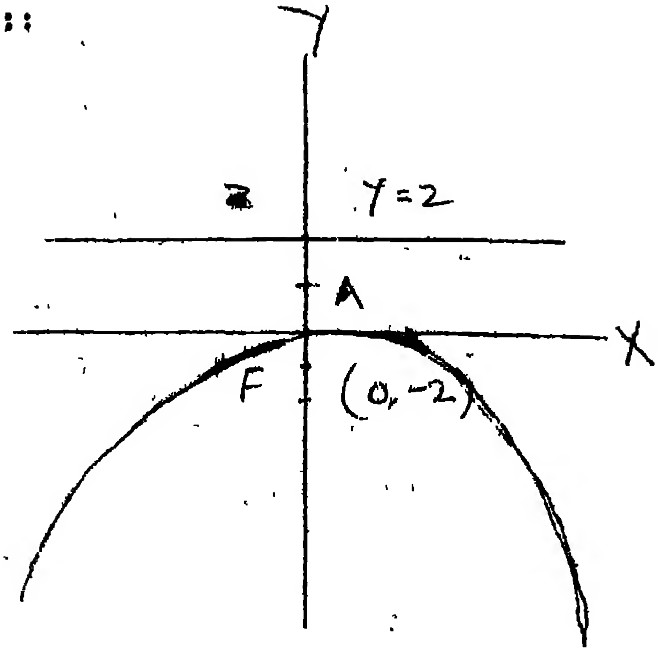


Fig. 12.13

Q. 2(c)

Let $y^2 = 4ax$ be the required equation. Since the parabola represented by it passes through $(2, 3)$, we have

$$9 = 8a$$

$$\Longrightarrow \quad a = \frac{9}{8}$$

$\therefore$ Required equation is

$$y^2 = 4\left(\frac{9}{8}\right) x$$

or $$y^2 = \frac{9}{2}\,x$$

$$\Longrightarrow \quad 2\,y^2 = 9x.$$

Q. 3(b)

Given equation is

$$y = -4x^2 + 3x$$

or $$y = -4\left(x^2 - \frac{3}{4}\,x\right)$$

or $\quad y = -4\left[x^2 - 2\cdot\frac{3}{8}x + \frac{9}{64}\right] + \frac{36}{64}$

or $\quad y = -4\left(x - \frac{3}{8}\right)^2 + \frac{9}{16}$

or $\quad \left(y - \frac{9}{16}\right) = -4\left(x - \frac{3}{8}\right)^2$

or $\quad \left(x - \frac{3}{8}\right)^2 = -\frac{1}{4}\left(y - \frac{9}{16}\right)$

or $\quad \left(x - \frac{3}{8}\right)^2 = -4\left(\frac{1}{16}\right)\left(y - \frac{9}{16}\right)$

Translating the origin to the point $\left(\frac{3}{8}, \frac{9}{16}\right)$ the transformed form of the given equation is

$$X^2 = -4\left(\frac{1}{16}\right) Y$$

With reference to the transformed axes,

Focus $F = \left(0, -\frac{1}{16}\right)$

Vertex $A = (0, 0)$

Equation to the directrix is

$$Y = \frac{1}{16}$$

Equation to the axis is

$$X = 0$$

$\therefore$ With reference to the original set of axes

Focus $F = \left(\frac{3}{8}, \frac{1}{2}\right)$

Vertex $A = \left(\frac{3}{8}, \frac{9}{16}\right)$

Equation to the directrix is

$$y - \frac{9}{16} = \frac{1}{16}$$

or $\quad y = \frac{5}{8}$

Equation to the axis is

$$x - \frac{3}{8} = 0$$

or $\quad x = \frac{3}{}$

Q.No.4 (b)

Vertex A = (0, 4)

Focus = (0, 2)

Directrix is $y = 6$..

Let P (x, y) be any point on the parabola

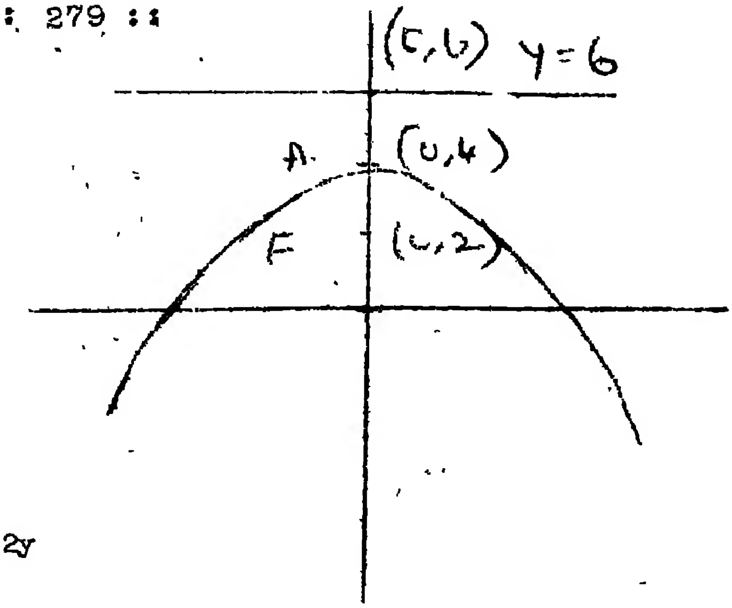


Fig 12.4

$$(x-0)^2 + (y-2)^2 = (6-y)^2$$

or $\quad x^2 - 4y + 4 = 36 - 12y$

or, $\quad x^2 = 32-8y.$

Alternative Method

Shifting the origin to the point (0, 4) and taking 11 set of axes of reference

Vertex A $\equiv$ (0, 0)

and Focus F $\equiv$ (0, -2)

According to new set of axes the equation to the parabola is

$$X^2 = -8Y$$

$\therefore$ According to original set of axes equation of the parabola

is $\quad x^2 = -8(y-4)$

or, $\quad x^2 = 32 - 8y.$

Q.No.4 (c)

Focus F $\equiv$ (-1, -2)

Directrix is $x - 2y + 3 = 0$

Let (x,y) be a point on the parabola

$\therefore$ From definition

$$(x+1)^2 + (y-2)^2 = \left(\frac{x-2y+3}{\sqrt{1+4}}\right)^2$$

or $$5\left[(x+1)^2 + (y-2)^2\right] = (x-2y+3)^2$$

which is required equation of the parabola.

Exercise 12.2

Q.No. (1) c

Equation of the given ellipse is

$$3x^2 + 2y^2 = 6$$

$$\Longrightarrow \frac{x^2}{2} + \frac{y^2}{3} = 1$$

Here foci will lie on y-axis.

Let e be the eccentricity.

$$\because \quad b^2 = a^2(1-e^2)$$

here $b^2 = 2$ and $a^2 = 3$

$$\therefore \quad 2 = 3(1-e^2)$$

or $$1-e^2 = \frac{2}{3} = 1 - \frac{1}{3}$$

$$e = \frac{1}{\sqrt{3}}.$$

$F_1 = (0, 1)$ and $F_2 = (0, -1)$

Length of major axis $= 2a$

$= 2\sqrt{3}.$

Length of minor axis $= 2b$

$= 2\sqrt{2}.$

Vertices are $(0, \pm\sqrt{3})$

Q.1 (d)

Given equation is

$$x^2 + 4y^2 - 2x = 0$$

or $$(x-1)^2 + 4y^2 = 1$$

or $$\frac{(x-1)^2}{1} + \frac{y^2}{\frac{1}{4}} = 1$$

Here $a^2 = 1$ and $b^2 = \frac{1}{4}$

Since $b^2 = a^2 (1-e^2)$

$$\frac{1}{4} = (1-e^2)$$

or $$1 - \frac{3}{4} = 1 - e^2$$

$$e = \frac{\sqrt{3}}{2}.$$

Shifting the origin to (1, 0) and taking a parallel set of axes of reference the transformed form of given equation is

$$\frac{x^2}{1} + \frac{y^2}{\frac{1}{4}} = 1.$$

Length of major axis $= 2a$

$= 2$

Length of minor axis $= 2b$

$= 2 \times \frac{1}{2} = 1.$

According to new set of axes

Foci $\equiv (\pm\, ae,\ 0)$

$\equiv \left(\pm \frac{\sqrt{3}}{2},\ 0\right)$

Vertices are $(\pm\, 1,\ 0)$

$\therefore$ According to original set of axes

Foci $= \left[1 \pm \frac{\sqrt{3}}{2},\ 0\right]$

Vertex $A_1 \equiv (2, 0)$ and $A_2 \equiv (0, 0)$

Q. 2(c) Foci at $(0, \pm 4) = (0, \pm ae)$

$\therefore \quad ae = 4$

$a \cdot \frac{4}{5} = 4$

$a = 5$

$\therefore \quad a^2 = 25$

$\because \quad b^2 = a^2 (1-e^2)$

$= 25\left(1 - \frac{16}{25}\right) = 25 \times \frac{9}{25}$

$b^2 = 9.$

$\therefore$ Equation of the ellipse is

$$\frac{x^2}{9} + \frac{y^2}{25} = 1.$$

or $\quad 25x^2 + 9y^2 = 225.$

Q. 2 (e) Foci at $(\pm 3, 0) = (\pm ae, 0)$

$\therefore \quad ae = 3 \qquad \text{———} (1)$

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the required equation

$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$

Since it passes through $(4, 1)$

$\therefore \quad \frac{16}{a^2} + \frac{1}{a^2(1-e^2)} = 1$

or $\quad \frac{16}{a^2} + \frac{1}{a^2-9} = 1$

or $\quad 16(a^2-9) + a^2 = a^4 - 9a^2$

Q.No.4 Let (x, y) be a point of the locus. Hence from the question

$$(x-0)^2 + (y-\ldots)^2 = \frac{4}{9}\left[9-y\right]^2$$

or $$9x^2 + 9(y-4)^2 = 4\left[81 + y^2 - 18y\right]$$

or $$9x^2 + 9y^2 + 144 - 7\cancel{2}y = 324 + 4y^2 - 7\cancel{2}y$$

or $$9x^2 + 5y^2 = 180$$

Exercise 12.3

Q.1 (c) Given equation is

$$\frac{x^2}{\frac{1}{3}} - \frac{y^2}{\frac{1}{2}} = 1$$

Length of the transverse axis $= 2a$

$$= 2/\sqrt{3} = \frac{2\sqrt{3}}{3}$$

and length of conjugate axis $= 2b$

$$= 2 \times \frac{1}{\sqrt{2}} = \sqrt{2}.$$

For eccentricity

$$b^2 = a^2(e^2 - 1)$$

$$\frac{1}{2} = \frac{1}{3}(e^2 - 1)$$

or $$\frac{3}{2} = e^2 - 1$$

or $$e^2 = \frac{5}{2}$$

$$e = \sqrt{5}/2$$

Foci $\left[\pm\sqrt{\frac{5}{6}}, 0\right]$

Vertices $= \left[\pm\frac{1}{\sqrt{3}}, 0\right]$.

Q.No.3

Obviously foci of the hyperbola are $(\pm 4, 0)$.

We know that the foci of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

are $(\pm ae, 0)$

$$\therefore \quad ae = 4 \qquad \text{------} \quad (1)$$

Further difference of the focal distances of any point on the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $2a$.

Here $2a = 2$ is given

$\Longrightarrow$ $a = 1$, $e = 4$.

Again $b^2 = a^2 (e^2-1)$ gives

$$b^2 = (16-1) = 15$$

Hence required locus is

$$\frac{x^2}{1} - \frac{y^2}{15} = 1$$

$$\Longrightarrow 15x^2 - y^2 = 15.$$

Q.No.4 The given equation

$$16x^2 - 3y^2 - 32x - 12y - 44 = 0$$

$$\Longrightarrow \frac{(x-1)^2}{3} - \frac{(y+2)^2}{16} = 1 \qquad \text{------} \quad (A)$$

Translation of axes to the point $(1, -2)$ transforms (A) into

$\frac{X^2}{3} - \frac{Y^2}{16} = 1$, which obviously represents a hyperbola.

For its eccentricity 'e' the formula

$b^2 = a^2 (e^2-1)$ gives

$$16 = 3 (e^2-1)$$

$$\Longrightarrow \frac{16}{3} + 1 = e^2$$

$$\Longrightarrow e^2 = \frac{19}{3}$$

$$\therefore e = \sqrt{\left(\frac{19}{3}\right)}$$

Length of transverse axis $= 2a$

$= 2.\sqrt{3}$

Length of conjugate axis $= 2b$

$= 2\sqrt{16}$

$= 8$

Exercise 12.4

Q.No. 2

We know that the line $y = mx + a/m$ touches the parabola $y^2 = 4ax$ at the point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Hence the line

$$lx + my + n = 0$$

$\Longrightarrow y = -\frac{l}{m}x - \frac{n}{m} =$ -, will touch the parabola if

$$-\frac{n}{m} = \frac{a}{-l/m}$$

$$\Longrightarrow \frac{n}{m} = \frac{am}{l}$$

$$\Longrightarrow am^2 = nl.$$

Q.No. 4

We know that the line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $c^2 = a^2m^2 + b^2$, at the point

$$\left(\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \frac{b}{\sqrt{a^2m^2+b^2}}\right)$$

Henco the line

$$lx + my + n = 0$$

$$\Longrightarrow \quad y = -\frac{l}{m}x - \frac{n}{m}$$ will touch the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ if

$$\left(-\frac{n}{m}\right)^2 = a^2\left(-\frac{l}{m}\right)^2 + b^2$$

or $$+\frac{n^2}{m^2} = \frac{a^2l^2}{m^2} + b^2$$

or $$+n^2 = a^2l^2 + b^2m^2$$

$$\Longrightarrow \quad a^2l^2 + b^2m^2 = n^2,$$

<u>Q.No.6</u> : <u>Solution</u>

Equation of the family of lines passing through $(a\cos\phi,\ b\sin\phi)$ is

$$y - b\sin\phi = m(x - a\cos\phi)$$

$$\Longrightarrow \quad y = mx + b\sin\phi - am\cos\phi \quad \text{------} \quad (1)$$

Since the condition for tangency of the line $y = mx + c$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$c^2 = a^2m^2 + b^2$$

Hence the condition for tangency of (1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $(b\sin\phi - am\cos\phi)^2 = a^2m^2 + b^2$

$$\Longrightarrow \quad m = -\frac{b}{a}\cot\phi$$

$\therefore$ Equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

at the point $(a \cos \phi , \; b \sin \phi)$ is

$$xy = -\frac{b}{a} \cot \phi (x - a \cos \phi)$$

$$\Longrightarrow \quad \frac{x \cos \phi}{a} + \frac{y \sin \phi}{b} = \cos^2 \phi + \sin^2 \phi$$

$$\Longrightarrow \quad \frac{x \cos \phi}{a} + \frac{y \sin \phi}{b} = 1$$

Alternative Method

Let P $(a \cos \phi , \; b \sin \phi)$ and

Q $(a \cos \phi_1 , \; b \sin \phi_1)$ be two points on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{—— (1)}$$

Equation to the chord P Q is

$$y - b \sin \phi = \frac{b (\sin \phi - \cos \phi_1)}{a(\cos \phi - \cos \phi_1)} (x - a \cos \phi)$$

$$\Longrightarrow \quad y - b \sin \phi = - \frac{b}{a} \cot \frac{\phi + \phi_1}{2} (x - a \cos \phi)$$

Since chord P Q $\longrightarrow$ tangent to the ellipse (1) at P,

if Q $\longrightarrow$ P.

$\therefore$ Equation of the tangent at the point P $(a \cos \phi , \; b \sin \phi)$

to the ellipse (1) is

$$y - b \sin \phi = - \frac{b}{a} \cot (x - a \cos \phi)$$

$$\Longrightarrow \quad \frac{y \sin \phi}{b} - \sin^2 \phi = - \frac{x \cos \phi}{a} + \cos^2 \phi$$

$$\Longrightarrow \quad \frac{x \cos \phi}{a} + \frac{y \sin \phi}{b} = 1.$$

Q.No.7 : Solution

Equation of the given ellipse is

$$4x^2 + 3y^2 = 5 \qquad \text{------ (1)}$$

Equation of the family of lines parallel to the given line

$y = 3x + 7$ is

$$y = 3x + \lambda \qquad \text{------ (2)}$$

Any member of the family of lines (2) will be a tangent, to the ellipse (1) if the points of intersection of (1) and (2) are real and coincident.

Eliminating y between (1) and (2) we have

$$31x^2 + 18\lambda x + (3\lambda^2 - 5) = 0 \qquad \text{------ (3)}$$

The condition for tangency of (2) to the ellipse (1) is,

$$(18\lambda)^2 - 4 \times 31 (3\lambda^2 - 5) = 0$$

$$\implies 81\lambda^2 - 93\lambda^2 + 155 = 0$$

$$\implies 12\lambda^2 = 155$$

$$\therefore \lambda = \pm\sqrt{\frac{156}{12}}$$

Required equations of the tangents are $y = 3x \pm \frac{1}{2}\sqrt{\frac{155}{3}}$

Points of contact $= \left[\mp \frac{3}{2}\sqrt{\frac{15}{31}}, \mp \frac{9}{2}\sqrt{\frac{15}{31}} \pm \sqrt{\frac{155}{12}}\right]$

$$= \left[\mp \frac{3}{2}\sqrt{\frac{15}{31}}\right]$$

Q. No. 8

Equation of the hyperbola is

$$4x^2 - 9y^2 = 1 \qquad \text{——} (1)$$

Family of lines parallel to the given line $4y = 5x + 7$ or,

$y = \frac{5}{4}x + \frac{7}{4}$ is

$$y = \frac{5}{4}x + \lambda \qquad \text{——} (2)$$

The line (2) will be a tangent to the hyperbola (1) if the roots of the equation

$$4x^2 - 9\left(\frac{5}{4}x + \lambda\right)^2 = 1$$

$\Longrightarrow$ $161x^2 + 360\lambda x + 16(1+9\lambda^2) = 0$, will be real and coincident.

The condition for coincident roots is

$$(360\lambda)^2 - 4 \times 161 \times 16(1 + 9\lambda^2) = 0$$

or $$(45\lambda)^2 - 161 - 161 \times 9\lambda^2 = 0$$

$\Longrightarrow$ $576\lambda^2 = 161$

or $$\lambda = \pm\frac{\sqrt{161}}{24}.$$

$\therefore$ Equation of the tangents required is

$$y = \frac{5}{4}x \pm \frac{\sqrt{161}}{24}.$$

or $$24y = 30x \pm \sqrt{161}.$$

Additional illustrations (Figures) to suppliment those given in the textbook.

Definition – Chords of a conic, passing through the focus of a conic are called Focal Chords.

Focal chords, which are perpendicular to the axis of the conic is called Latus Rectum.

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It is worthy to note that except parabola, Ellipse and hyperbola have a pair of latus-recta.

The focal chord LL^1, which is perpendicular to the axis of the parabola, is the latus-rectum of the parabola.

Consider the parabola $y^2 = 4ax$ having $(a, 0)$ its focus. As parabola is symmetrical about its axis, hence $FL = FL^1$.

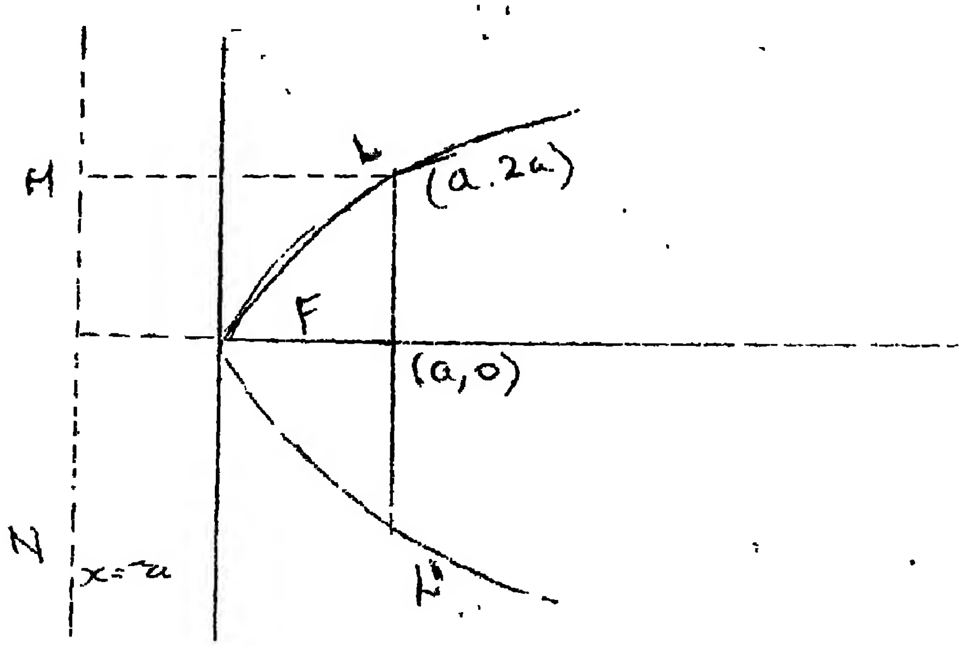

Fig. 12.15

Q. No. 8

Equation of the hyperbola is

$$4x^2 - 9y^2 = 1 \qquad \text{--- (1)}$$

Family of lines parallel to the given line $4y = 5x + 7$ or,

$y = \frac{5}{4}x + \frac{7}{4}$ is

$$y = \frac{5}{4}x + \lambda \qquad \text{--- (2)}$$

The line (2) will be a tangent to the hyperbola (1) if the roots of the equation

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or $$(45\lambda)^2 - 161 - 161 \times 9\lambda^2 = 0$$

$\implies$ $576\lambda^2 = 161$

or $$\lambda = \pm\frac{\sqrt{161}}{24}.$$

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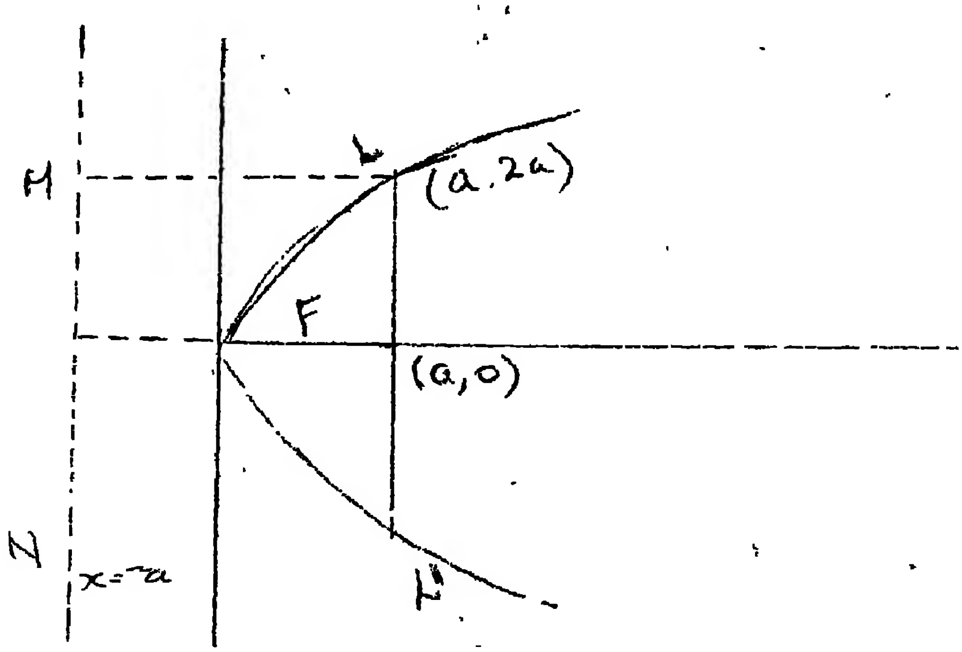


Fig. 12.15

Let LM be perpendicular on the directrix of the parabola.

From definition of parabola.

$$FL = LM = 2a$$

$$\therefore \quad L \equiv (a, 2a) \text{ and } L^1 \equiv (a, -2a)$$

Length of latus rectum = 4a.

Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose major axis lie along x-axis. Its foci F_1 and F_2 are $(-ae, 0)$ and $(ae, 0)$.

Two latus-recta of the ellipse are $L_1L^1_1$ and $L_2L'_2$.

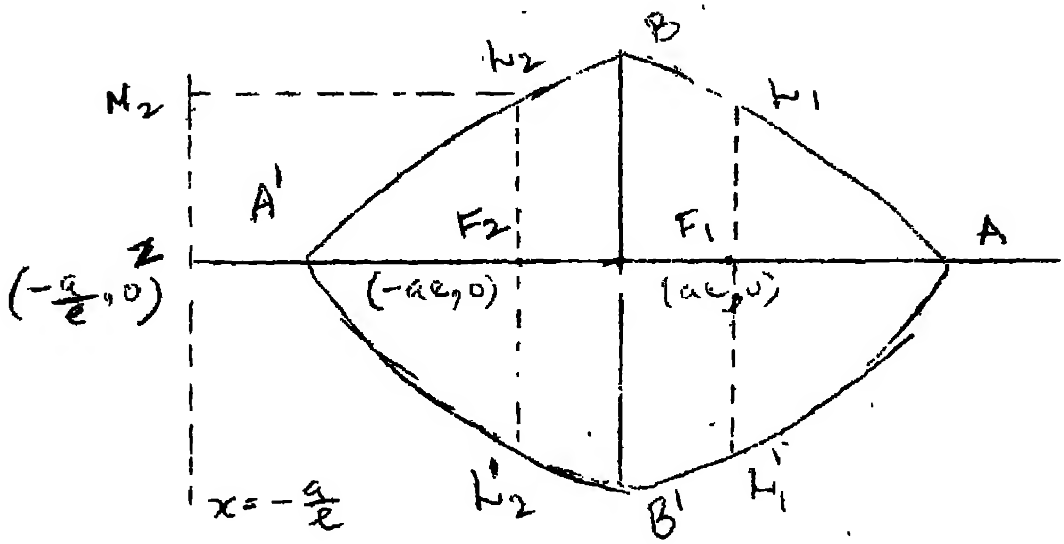


Fig. 12.16

As ellipse too is symmetrical about its axes :

$$\therefore \quad F_2L_2 = F_2L'_2$$

$$F_2L_2 = e.\, L_2M_2 = e\left[\frac{a}{e} - ae\right]$$

$$= a(1 - e^2) = \frac{a^2(1-e^2)}{a}$$

$$= \frac{b^2}{a}$$

$\therefore L_2 \equiv (-ae, \frac{b^2}{a})$

and $L'_2 \equiv (-ae, -\frac{b^2}{a})$

Similarly $L_1 \equiv (ae, \frac{b^2}{a})$ and $L'_1 = (ae, -\frac{b^2}{a})$

$\therefore$ Length of both the latus-recta of the ellipse $= \frac{2b^2}{a}$.

Similar statement will clarify the students that the length of latus-recta of the hyperbola $= \frac{2b^2}{a}$ and co-ordinates of the ends of latus-recta are $(\pm ae, \pm \frac{b^2}{a})$.

CHAPTER TEST

(a) Oral Test

(i) Find the focus of the parabola $y^2 = 8x$ Ans: (2,0)

(ii) Find the equation of the directrix of the parabola $y^2 = -12x$. Ans: $x = 3$.

(iii) Find the equation of the axis of the conic $x^2 = 6y$.

(iv) Find the length of the major axis of the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

Ans: 8

(v) Find eccentricity of the ellipse

$$\frac{x^2}{36} + \frac{y^2}{25} = 1$$

Ans: $e = \frac{\sqrt{11}}{6}$

(vi) Find the eccentricity of the hyperbola

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Ans: $e = 5/4$

(vii) Find the foci of the hyperbola

$$\frac{x^2}{25} - \frac{y^2}{9} = 1$$ Ans : $\left[\pm\sqrt{34},\ 0\right]$

(viii) Find the foci of the hyperbola

$$\frac{y^2}{9} - \frac{x^2}{4} = 1$$ Ans: $\left[0,\ \pm\sqrt{13}\right]$

(b) Written Test

(i) Find the equation of the locus of such a point, which is the middle point of any chord of the parabola

$$y^2 = 4\,ax$$ Ans: $y^2 = 2\,ax$.

(ii) Find the eccentricity of the conic

$$x^2 - 2x - 4y^2 = 0$$ Ans : $\left[e = \frac{\sqrt{5}}{2}\right]$

(iii) If $(at^2, 2at)$ and $(at_1^2, 2at_1)$ are the ends of a focal chord of the parabola, prove that $tt_1 = -1$.

(iv) Show that $y - x - 2 = 0$ is a tangent to the parabola $y^2 = 8x$. Find the point of contact. Ans : $(2, 4)$

(v) Find the vertex, axis, latus return and focus of the parabola, $y^2 = 4y - 4x$.

Ans: Vertex $(0, 2)$
Axis $y = 2$
latus return $= 4$
focus $(-1, 2)$

References: As given in the chapter X, on circles and family of circles.

CHAPTER - 13

TRIGONOMETRIC FUNCTIONS

1. Introduction

The word trigonometry is derived from three Greek words - 'tri', means three, 'gono' means angles and 'metry' means measure and hence the literal meaning is 'measurement of the triangle'.

In trigonometry we have a good deal of combination of algebra and geometry. There are algebraic symbols, formulae and equations which make the subject more interesting and useful for practical applications. To state a few, it is useful in measuring height of the mountains, the summits which can not be reached, the distance of inaccessible objects, the width of rivers without undertaking the trouble of actually crossing them etc. It is rather indispensable for industrial engineering, surveying, navigation, astronomy, seismology and the study of occurance of sunspot and so on.

Method of solving triangles originated from astronomy and for a long time trigonometry developed as a branch of astronomy. The methods for solving (spherical) triangles were recorded for the first time by the Greek astronomer Hipparchus in the middle of the 2nd Century B.C.

The Greek astronomers did not deal in Sines, Cosines and tangents. In place of tables of these quantities they used tables that permitted to find the chord of a circle from the intercepted arc. which were measured in degrees and minutes, chords too were measured in degrees (one degree being one sixtieth of the radius), minutes and seconds.

Arya Bhatt in 511 A.D. used sine & cosine as 'ज्या' and 'कोज्या' in astronomical calculations. Medieval astronomers of India also made considerable advances in Trigonometry.

In 15th Century German astronomer Johann Muller (1436-1476) better known under the name Regiomontanus rediscovered his theorems.

Approximations of values of sin θ, Cos θ, are credited to Newton and Leibnitz. But Keralite mathematicians used such approximations at least two centuries earlier.

The knowledge of trigonometrical functions will be helpful in finding solution of trigonometrical equations, solution of triangles, problems related to heights and distances, inverse trigonometrical functions etc.

Ancient Definition of Trigonometrical terms:

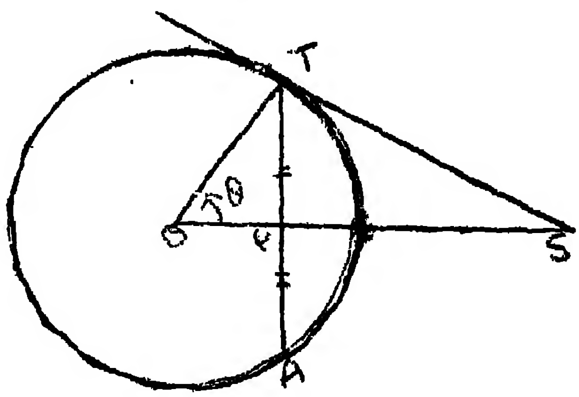


Fig. 13.1

Consider a circle with Centre O and unit radius OT. AT is a chord of this circle

OS bisects AT at P

Let $\angle POT = \theta$

Tangent TS intersects OP extended at S

Half of the chord AT is PT

Now $\frac{PT}{OT} = \frac{PT}{1} = PT = \sin\theta$

the length PT is considered as $\sin\theta$.

As OT is of unit length OS is sec θ, and TS and tan θ.

Other trigonometric ratios Cos θ, Cosec θ and cot θ can be derived from the sin θ, sec θ and tan θ.

Later on these trigonometric ratios based on parts of a unit circle are generalised as ratio by considering them as corresponding parts of a circle of any radius.

<u>Radian Measure:</u>

With the help of unit circle co-ordinate of Point P(x,y) is given as

$x = \text{Cos}\,\theta$

and $y = \text{Sin}\,\theta$

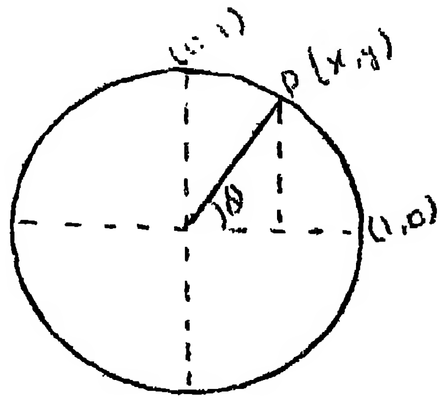


Fig. 13.2

Teacher should explain that as P rotates from one quadrant to another quadrant the values of x,y changes and so the values of trigonometric ratios. At this stage Teacher should also stress the importance of radian measure and help the student in deriving value of 1^c in degree and value of 1^o in radian

$$\text{Since} = \frac{2\pi r}{r} = 2\pi^c$$

and therefore measure of $1^c = \frac{180}{\pi} = 57^o, 16'$ approximately

also $1^o = \frac{\pi}{180}$ radian = 0.01746 radian approximately

Teachers should give proper practice in converting radian measure to degree and degree measure to radian.

Content Analysis:

In this section the number of each sub section is in accordance with the text book.

13.2 Angle and their measurements:

The anti clockwise direction for the rotation of initial side is termed as positive angle.

Measurement of Angles

1. Sexagesimal System (British System)

1 right angle	=	90^{o}
1 degree	=	60 minutes = 60'
1 minute	=	60 seconds = 60"

2. Centesimal System (French System)

1. right angle	=	100 grades = 100^{g}
1 grade	=	100 minutes = 100'
1 minute	=	100 seconds = 100"

3. Circular System

The radian is defined as an angle subtended at the centre of a circle by an arc which is equal in length with the radius of the circle.

Length of the arc	Radius	Radian
r	r	1^{c}
S	r	$\frac{S}{r}^{c}$
$2\pi r$	r	2π

2π radian = 360^{o} and π radian = 180^{o}

The teacher should explain the definition of angle and system of their measurement. He should derive fundamental trigonometrical results with the help of diagrams.

In exercise 13.1, question 8, student are required to find the angle between the hour and minute hand at 7.20, They usually take it as 90°. But it is not so. At 7 O'clock the hour hand will be at 7 & at 7.20 the hour hand must have moved through some angle. The angle can be calculated as follows:-

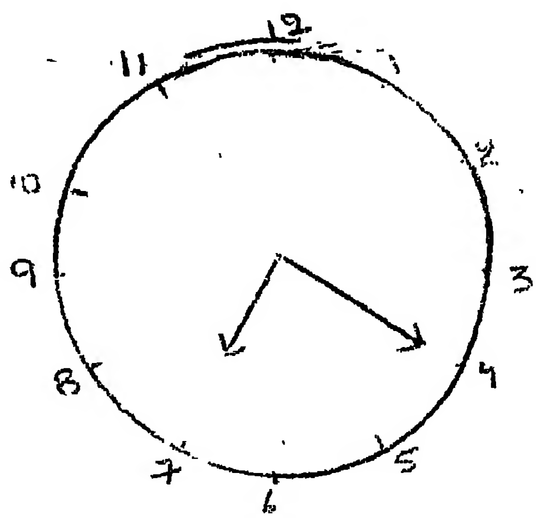


Fig. 13.3

In 60 minutes the hour hand traces 30° so, in 20 minutes the hour hand moves $\left(\frac{30 \times 20}{60}\right)^\circ = 10^\circ$. Thus angle between the two hands $= 90^\circ + 10^\circ = 100^\circ$.

13.3 Circular Functions:

In the article 13.3 of the text book the circular functions are defined with the help of definition of Sin θ, Cos θ, tan θ, cot θ, sec θ, cosec θ, find that

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

There are some functions $f(x)$, such that $f(-x)=f(x)$ for all x.

and there are some functions:

$f(-x) = -f(x)$ for all x.

To differentiate between these two type of functions, they are termed as even and odd functions respectively.

Ex. <u>Even functions</u> $f(x) = x^2+1$

$f(x) = x^4+x^2+2$

$f(x) = \sin^2 x$

<u>Odd functions:</u>

$f(x) = x$

$f(x) = x^3+x$

There are some function like

$f(x) = x^2+x$

$f(x) = x^4-2x$

in which $f(-x) = x^2-x$ or $f(-x) = x^4+2x$

is neither $f(-x) = +f(x)$ for all values of x

nor $f(-x)$ $= -f(x)$ for all values of x, they are termed neither even nor odd.

If a function f is periodic then the smallest positive value of T if it exists such that $f(x+T) = f(x)$ for all x is called the period of the function.

At this stage the teacher has to point out that a constant function is not periodic even though $f(x+T)=f(x)$ since there is no smallest $T>0$ exist which will make this true.

Example:

$$f(x) = 2$$

$$f(x+T) = 2$$

$$\therefore f(x+T) = f(x)$$

Here T is not a period because however small we take T, we can always find a value smaller than this value.

∴ All periodic functions are assumed to be non constant functions.

After introducing the periodicity of the trigonometric functions, the following formula can be introduced.

$\sin(2n\pi + \theta) = \sin\theta$, where n is an integer. This is applicable for all trigonometric ratios. This formula is useful for finding the values of the trigonometric functions for larger values of θ. Some more examples may be given by the teacher.

Signs of trigonometric ratios:

To explain the signs of all trigonometric ratios in the four quadrants, the teacher should draw different diagrams, placing the second arm of the angle in different quadrants.

II quadrant

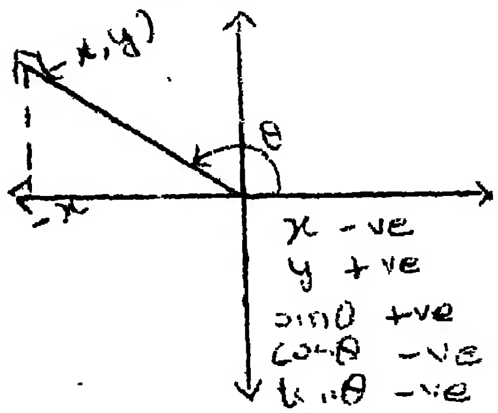


Fig. 13 5

I quadrant

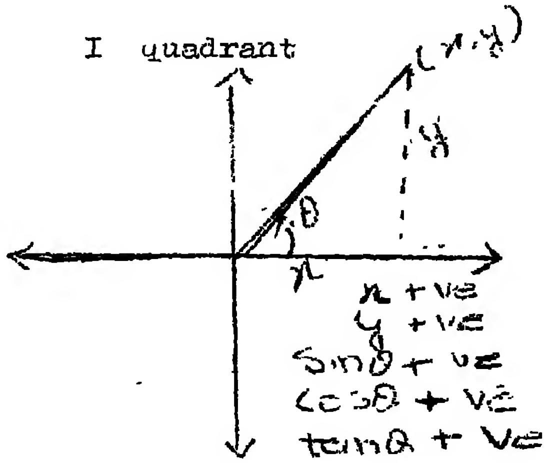


Fig. 13.4

III quadrant

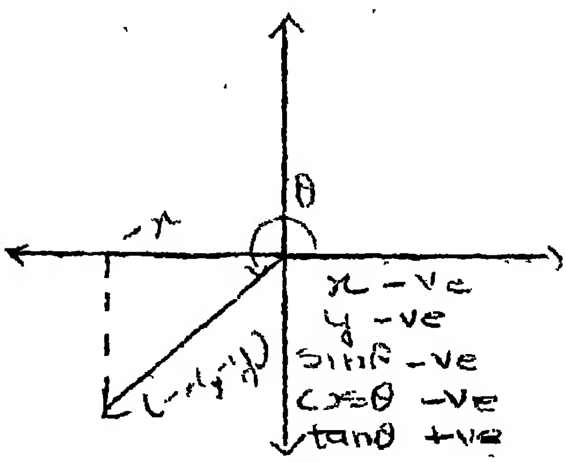


Fig. 13.6

IV quadrant

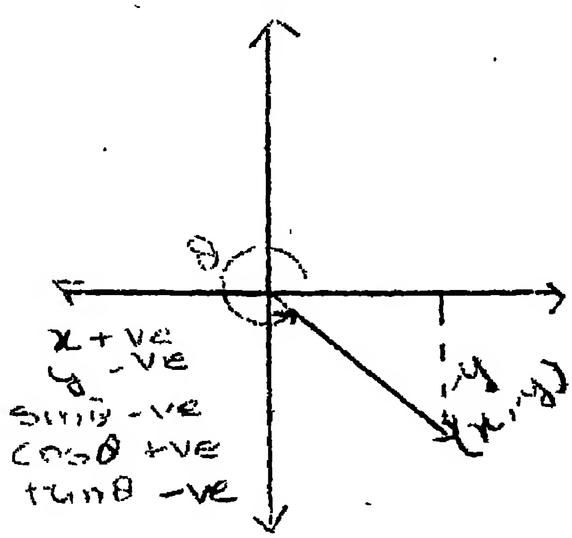


Fig. 13.7

The same rule is applicable for other complementry ratios

The fact that $\sin\theta$ lies between -1 and $+1$ can be explained with the help of a diagram.

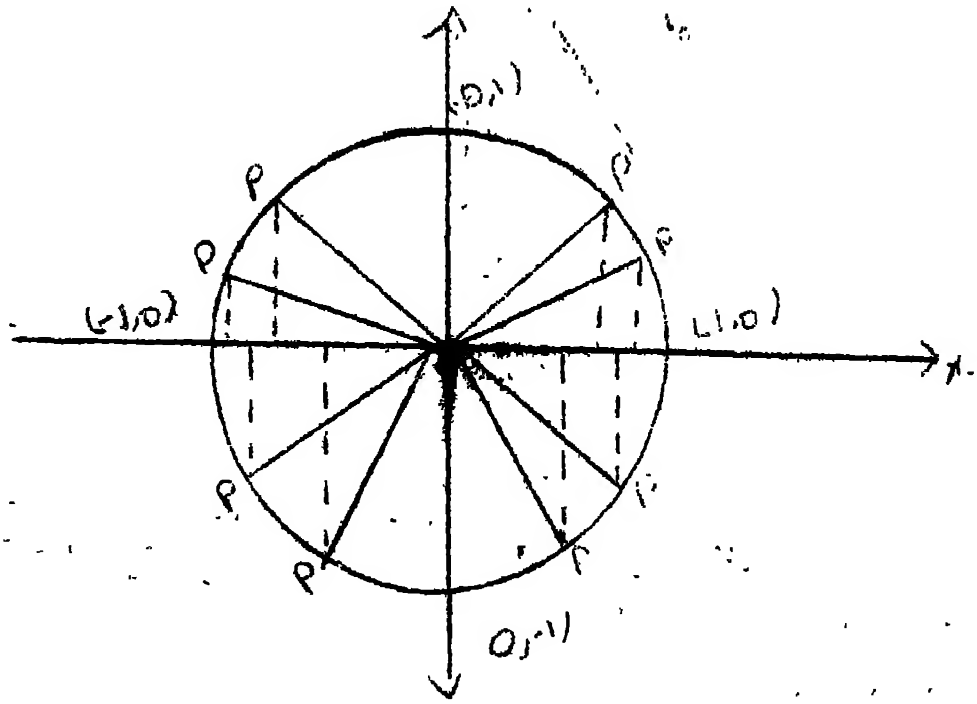


Fig. 13.8

13.4 Trigonometric Identities

The teacher may give more examples to bring out the difference between a trigonometric identity and a trigonometric equation.

Example 13.3 Alternative method

Hint : Since θ is in the 3rd quadrant draw the following diagram.

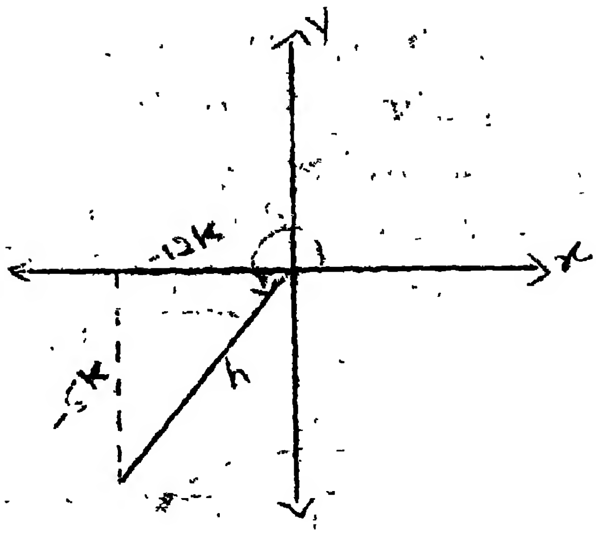


Fig. 13.9

From the diagram. We can find r and then other trigonometrical ratios.

The teacher should explain the following point by means of the diagram given below. Since it is required for the derivation of Cos (A-B)

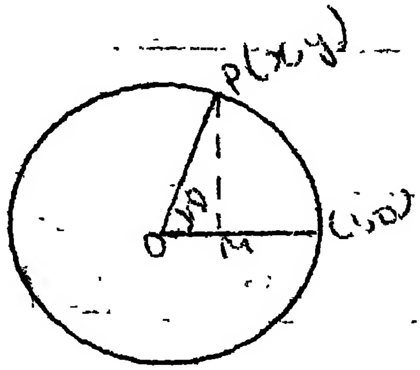


Fig: 13.10

$\cos\theta = \frac{OM}{OP} = OM = x$

$\sin\theta = \frac{PM}{OP} = PM = y$

Any point taken on the unit circle will be of the form $(\cos\theta, \sin\theta)$.

13.5 Cosine of sum of Two angles:

The derivation of Cos (A+B) and Sin(A+B) are dealt with the help of the diagram given on the right hand side.

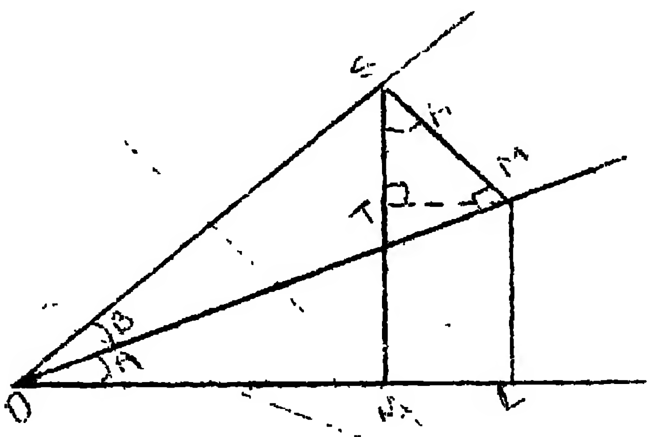


Fig. 13.11

MOL = A and MOS = B

So, Cos (A+B) $= \frac{OR}{OS}$ (by deffinition)

$= \frac{OL-RL}{OS} \quad \frac{OL-TM}{OS} \quad \because RL=TM$

$= \frac{OL}{OS} \quad \frac{TM}{OS}$

$= \frac{OL}{OM} \cdot \frac{OM}{OS} \quad \frac{TM}{SM} \cdot \frac{SM}{OS}$

$=$ Cos A. Cos B − SinA. Sin B,

and Sin (A+B) $= \frac{SR}{OS} = \frac{TR+TS}{OS} = \frac{ML}{OS} + \frac{TS}{OS}$

$= \frac{ML}{OM} \cdot \frac{OM}{OS} + \frac{TS}{SM} \cdot \frac{SM}{OS}$

= SinA. CosB + Cos A. Sin B

<u>Cosine and Sine of Difference of Two angles</u>

The formula for Cos (A-B) and Sin (A-B) can be derived with the help of the diagram.

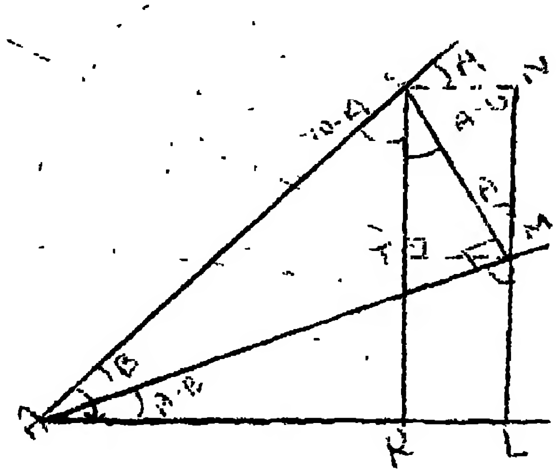


Fig. 13.12 .

$$\cos(A-B) = \frac{OL}{OM} = \frac{OR+RL}{OM}$$

$$= \frac{OR}{OS} \cdot \frac{OS}{OM} + \frac{TM}{SM} \cdot \frac{SM}{OM}$$

$$= \cos A \; \cos B + \sin A \cdot \sin B$$

$$\sin(A-B) = \frac{ML}{OM} = \frac{RS - TS}{OM}$$

$$= \frac{RS}{OS} \cdot \frac{OS}{OM} - \frac{TS}{SM} \cdot \frac{SM}{OM}$$

$$= \sin A \cdot \cos B - \cos A \cdot \sin B$$

To derive Sine of the difference of two angles:

Refer figure 13.10 in article 13.5 of the textbook.

We see that angles $P_1 O P_2$ and $P_o O P_3$ equal in length.

Hence distance formula is used. <u>Corrollary - 1</u>

1. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B.$
2. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B.$

By using the product formula and taking $A+B=C$ and $A-B=D$ and deriving the values of $A=\frac{C+D}{2}$ and $B=\frac{C-D}{2}$. We can derive the relations.

1. $\sin C + \sin D = 2\sin\frac{C+D}{2} \cdot \sin\frac{C-D}{2}$
2. $\sin C - \sin D = 2\cos\frac{C+D}{2}\ \sin\frac{C-D}{2}$
3. $\cos C + \cos D = 2\cos\frac{C+D}{2} \cdot \cos\frac{C-D}{2}$
4. $\cos C - \cos D = 2\sin\frac{C+D}{2}\ \sin\frac{D-C}{2}$

After deriving product and sum formulae as given in the text book, teacher should give exercises involving these results. Teacher should derive sum and product formula for other trigonometric functions. These are already stated in the book before exercise 13.3 in the section 13.5. In this section with the help of these formulae method for derivation of values for sine and cosine of any angle is also dealt with.

$$\sin 15^\circ = \sin(45^\circ - 30^\circ)$$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}}\ \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

and $\cos 15^\circ = \cos (45^\circ - 30^\circ)$

$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$

$= \frac{\sqrt{3} + 1}{2\sqrt{2}}$

If we have to determine sin $22\frac{1}{2}^\circ$ and Cos $22\frac{1}{2}^\circ$ we derive as follows:

$2 \sin^2\theta = 1 - \cos 2\theta$ if $\theta = 22\frac{1}{2}^\circ$

$\sin^2 22\frac{1}{2}^\circ = \frac{1}{2}\left[1 - \cos 45^\circ\right]$

or $\sin^2 22\frac{1}{2}^\circ = \frac{1}{2}\left[1 - \frac{1}{\sqrt{2}}\right]$

$= \frac{1}{2}\left[\frac{\sqrt{2} - 1}{\sqrt{2}}\right] \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{2 - \sqrt{2}}{4}$

$\therefore \sin 22\frac{1}{2}^\circ = \frac{1}{2}\sqrt{2 - \sqrt{2}}$

Similarly

$\cos^2 22\frac{1}{2}^\circ = \frac{1}{2}\left[1 + \cos 45^\circ\right]$

$= \frac{1}{2}\left[1 + \frac{1}{\sqrt{2}}\right]$

$= \frac{1}{2} \frac{\sqrt{2} + 1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{1}{4} \cdot 2 + \sqrt{2}$

$\therefore \cos 22\frac{1}{2}^\circ = \frac{1}{2}\sqrt{2 + 2}$

Also :

$\sin 105^{\circ} = \sin(90 + 15^{\circ})$

$= \cos 15^{\circ} = \frac{\sqrt{3}+1}{2\sqrt{2}}$.

$\cos 105^{\circ} = \cos(90 + 15^{\circ})$

$= -\sin 15^{\circ}$

$= -\frac{\sqrt{3}-1}{2\sqrt{2}}$

13.6 Table of Trigonometric functions:

In the article 13.3 of the Textbook values of the trigonometric functions for different angles are derived with the help of the figure 13.6, 13.7 and 13.8. In this way we can develop the table for values of trigonometric functions for different angles variying from 0° to 180°.

	0°	30°	45°	60°	90°	120°	135°	150°	180°
Sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
Cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
Tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
Cot	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	not defined
Sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1
Cosec	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined

By the help of these values and with the help of different trigonometrical relations we can derive values of different trigonometric ratios for any angle. We can find out the values of trigonometric ratios with the help of tables also given at the end of the text book. Derivation of these values are deal in the examples 13.15, 13.16 and 13.17 of the text book.

13.7 Graphs of Trigonometric Functions

For drawing the graphs of the functions of the type $\sin ax$, $\sin (ax+b)$ and $C \sin (ax+b)$ it can be found that the period of each of these functions is $\frac{2\pi}{a}$.

Verification

(1) $f(x) = \sin ax$

$$f\left(x + \frac{2\pi}{a}\right) = \sin a\left(x + \frac{2\pi}{a}\right)$$

$$= \sin (ax + 2\pi)$$

$$= \sin ax$$

$$= f(x)$$

Hence the period is $\frac{2\pi}{a}$

) $f(x) = 5 \sin (3x + 4)$

$$f\left(x + \frac{2\pi}{3}\right) = 5 \sin \left[3\left(x + \frac{2\pi}{3}\right) + 4\right]$$

$$= 5 \sin \left[(3x + 2\pi) + 4\right]$$

::

$$= 5 \sin \left[(3x + 4) + 2\pi \right]$$

$$= 5 \sin (3x+4)$$

$$= f(x)$$

The period is $\frac{2\pi}{3}$

13.8 Conditional Identities:

Teacher should make it clear the difference between Identities and conditional identities. By giving examples he must clarify that an Identity is true for all values of θ While conditional identities are true only under certain imposed conditions. e.g. $A+B+C = \pi$, $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$ etc.

When $A + B + C = \pi$

$$\sin (A+B) = \sin (\pi - C)$$

$$= \sin C$$

$$\cos (A+B) = \cos (\pi - C)$$

$$= -\cos C$$

$$\tan (A+B) = \tan (\pi - C)$$

$$= -\tan C$$

if $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$

Then $\sin\left(\frac{A+B}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cos\frac{C}{2}$

$\cos\left(\frac{A+B}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{C}{2}\right) = +\sin\frac{C}{2}$

$\tan\left(\frac{A+B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cot\frac{C}{2}$

13.9 Trigonometric Equations

Before taking up the solution of

$\sin\theta = \sin\alpha$

$\cos\theta = \cos\alpha$

and $\tan\theta = \tan\alpha$

The teacher should explain the solution of

$\sin\theta = 0$, i.e. $\sin\theta = \sin 0$ or $\sin n\pi$

$\cos\theta = 0$, i.e. $\cos\theta = \cos\frac{\pi}{2}$ or $\cos n\frac{\pi}{2}$

$\tan\theta = 0$, i.e. $\tan\theta = \tan 0$ or $\tan n\pi$

where n is an integer and thus we get value of θ in each case as below:

$\theta = 0, \pm\pi, \pm 2\pi, 3\pi, \ldots\ldots\ldots\ldots \pm n\pi$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots\ldots\ldots (2n+1)\frac{\pi}{2}$

$\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots\ldots\ldots\ldots \pm n\pi.$

Where n is an integer.

After this teacher may take examples of Trigonometric equations like -

(a) $\sin\theta = \frac{1}{2}$

$= \sin 30$

$= \sin \frac{\pi}{6}$

$\therefore \quad = \frac{\pi}{6}$

$= n\pi + (-1)^n \frac{\pi}{6}$; where n is integer

(b) $\cos = \frac{1}{2} \doteq \cos \frac{\pi}{3}$ or $\cos\left(-\frac{\pi}{3}\right)$

$\theta = \frac{\pi}{3}$ or $-\frac{\pi}{3}$

or $\theta = 2n\pi + \frac{\pi}{3}$ where n is integer.

Further, teacher may deal with the solution of Trigonometric equations of the type -

$\sin\theta = \sin\alpha$

$\cos\theta = \cos\alpha$

and $\tan\theta = \tan\alpha$

the solutions of these Trigonometric equations are already dealt in section 13.9 of the book.

Solution of some of the questions from different exercises are given below:

Exercise 13.3

Q.16 Find $\sin 7\frac{1}{2}^\circ$

$$\cos 15^\circ = 1-2\sin^2 7\tfrac{1}{2}^\circ$$

$$\sin^2 7\tfrac{1}{2}^\circ = \frac{1-\cos 15^\circ}{2}$$

$$= \frac{1-\left(\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\right)}{2}$$

$$= \frac{2\sqrt{2}-\sqrt{3}-1}{4\sqrt{2}}$$

$$= \frac{(2\sqrt{2}-\sqrt{3}-1)\sqrt{2}}{4\sqrt{2}\sqrt{2}}$$

$$= \frac{4-\sqrt{6}-\sqrt{2}}{8}$$

$$\therefore \sin 7\tfrac{1}{2}^\circ = \frac{\sqrt{4-\sqrt{2}-\sqrt{6}}}{2\sqrt{2}}$$

Q.No. 18

Prove that $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5\pi}{16}$

$$\text{LHS} = \frac{1}{4}\left[\left(2\sin\frac{4\pi}{5}\sin\frac{\pi}{5}\right)\left(2\sin\frac{3\pi}{5}\sin\frac{2\pi}{5}\right)\right]$$

$$= \frac{1}{4}\left[\cos\frac{3\pi}{5}-\cos\pi\right]\left[\cos\frac{\pi}{5}-\cos\pi\right]$$

$$= \frac{1}{4}\left[\cos 108^\circ + 1\right]\left[\cos 36^\circ + 1\right]$$

$$= \frac{1}{4}\left[\cos(90^\circ+18^\circ)+1\right]\left[\cos 36^\circ + 1\right]$$

$$= \frac{1}{4}\left[-\sin 18^\circ + 1\right]\left[\cos 36^\circ + 1\right]$$

$$= \frac{1}{4}\left[\frac{\sqrt{5}-1}{4}+1\right]\left[\frac{\sqrt{5}+1}{4}+1\right]$$

$$= \frac{1}{4}\left[\frac{-\sqrt{5}+1+4}{4}\right]\left[\frac{\sqrt{5}+1+4}{4}\right]$$

$$= \frac{1}{64}\left[5-\sqrt{5}\right]\left[5+\sqrt{5}\right]$$

$$= \frac{1}{64}\left[25-5\right] = \frac{20}{64} = \frac{5}{16} \quad \text{R.H.S.}$$

Q.No. 21

$$\cos^2 A + \cos^2(A+120^\circ) + \cos^2(A-120^\circ) = \frac{3}{2}$$

$$\text{L.H.S} = \cos^2 A + \cos^2(A+120^\circ) + 1 - \sin^2(A-120^\circ)$$

$$= 1+\cos^2 A + \left[\cos^2(A+120^\circ) - \sin^2(A-120^\circ)\right]$$

$$= 1+\cos^2 A + \cos(A+120^\circ + A-120^\circ)\cos(A+120^\circ - A+120^\circ)$$

$$= 1 + \cos^2 A + \cos 2A \cos 240^\circ$$

$$= 1 + \cos^2 A + \cos 2A \,.\, \cos(180^\circ + 60^\circ)$$

$$= 1 + \cos^2 A + \cos 2A\,(-\cos 60^\circ)$$

$$= 1 + \cos^2 A + (2\cos^2 A - 1)\left(-\frac{1}{2}\right)$$

$$= 1 + \cos^2 A - \cos^2 A + \frac{1}{2} = \frac{3}{2} = \text{R.H.S.}$$

Q.No.24. Prove that $\cos 6A = 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1$

L.H.S.= $\cos(4A + 2A)$

$= \cos 4A \cos 2A - \sin 4A \sin 2A$

$= (2\cos^2 2A - 1)\cos 2A - 2\sin 2A \cos 2A \sin 2A$

$= 2\cos^3 2A - \cos 2A - 2\sin^2 2A \cos 2A$

$= 2\left[2\cos^2 A - 1\right]^3 - (2\cos^2 A - 1) - 2\cos 2A\,(1 - \cos^2 A)$

$= 2\left[8\cos^6 A - 12\cos^4 A + 6\cos^2 A - 1\right]$

$-2\cos^2 A + 1 - 2(2\cos^2 A - 1)\left[1 - (2\cos^2 A - 1)^2\right]$

$= 16\cos^6 A - 24\cos^4 A + 12\cos^2 A - 2 - 2\cos^2 A + 1$

$-(4\cos^2 A - 2)(-4\cos^4 A + 4\cos^2 A)$

$= 16\cos^6 A - 24\cos^4 A + 10\cos^2 A - 1$

$-(-16\cos^6 A + 16\cos^4 A + 8\cos^4 A - 8\cos^2 A)$

$= 16\cos^6 A - 24\cos^4 A + 10\cos^2 A - 1$

$+16\cos^6 A - 24\cos^4 A + 8\cos^2 A$

$= 32\cos^6 A \quad 48\cos^4 A + 18\cos^2 A - 1 =$ R.H.S.

3. Learning Outcomes

(a) Essential learning out comes for all.

After studying this chapter student should be able to:

i) Explain angles and the way they are measured.

ii) Convert measurement of angle from one system to another.

iii) Establish relationship among trigonometric ratios.

iv) discriminate between trigonometric identities and equations.

v) draw graph of trigonometric functions.

vi) tell values of trigonometric ratios for certain angles.

vii) calculate values of trigonometric functions for different angles with the help of known results.

viii) solve trigonometric identities, equations and derive relation ship between trigonometric ratios.

(b) Learning Outcomes for the higher group:

i) Student should be able to derive and solve different trigonometric relations and difficult questions of trigonometric identities.

ii) Teacher should help the students of higher group so that they are able to solve difficult type of conditional identities.

iii) Students of the higher group should be able to trace the graphs involving trigonometric functions.

iv) Students of the higher group should be able to solve difficult questions of trigonometric equations and find their general values.

4. <u>Teaching Strategies</u>

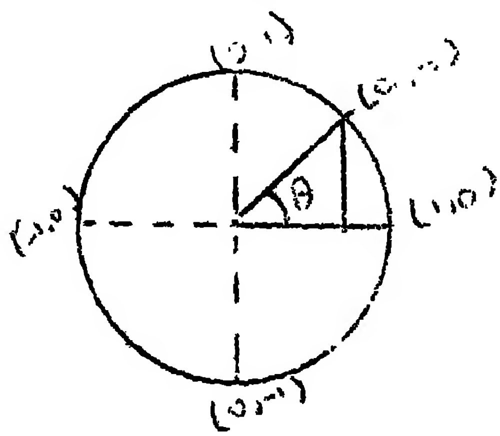
<u>Motivation:</u>

Teacher should introduce each concept with reference to learners' previous knowledge.

In the begining teacher should use proper diagrams and teaching aids. The concept of angle and mode of their measurement are dealt with reference to historical development in measurement; British system of measurement french system of measurement & Radian measure of measurement Should be discussed with the student with practical examples of inter conversion from one system to another system.

(ii) <u>Trigonometric Functions:</u>

(a) Trigonometric functions are introduced to student with the help of circle of unit radius.



In the diagram a circle with unit radius is given. The co-ordinate of point P is (a,b) and distance OA=OP=OB=1. By definition student can easily derive that

$\sin\theta = b$ and $\cos\theta = a$.

Here teacher should explain that a is cosine of θ, read as cosine theta or cos θ, and sine of θ read as sine theta or sin θ. So is the case with tangent θ or tan θ.

(b) Teacher should ask the child to form the angle AOB

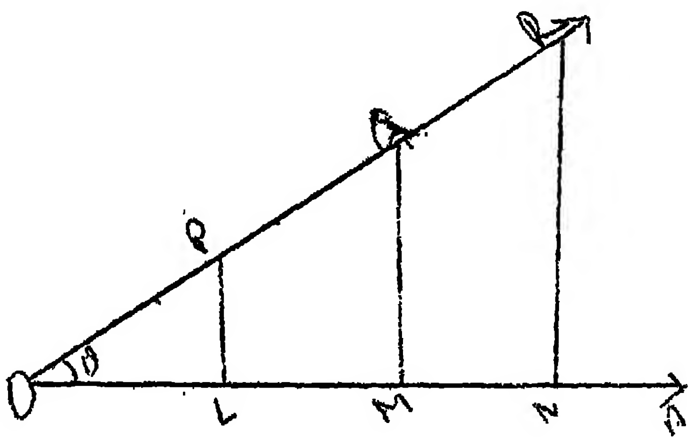
i) then select point P, Q, R....on OB

ii) draw ⊥ from P, Q, R...on OA

measure length of each.

⊥ PL, QM & RN

also measure length of each base OL, OM & ON & distances OP, OQ, OR.



Now teacher should ask the student to find out different ratio's among the sides of right triangle OPL, OQM and ORN.

From this ask the child to conclude that

i) What will be the ratio of perpendicular to hypotenuse in each right triangle, is it equal in all cases ?

ii) What will be the ratio of base to hypotenuse in each right triangle, is it equal in all cases ?

iii) What will be the ratio of perpendicular to base in each right triangle, is it equal in all cases ?

. What will you conclude ? Here teacher should explain that

$$\frac{\text{perpendicular}}{\text{Hypotenuse}} = \text{Sin } \theta$$

$$\frac{\text{Base}}{\text{Hypotenuse}} = \text{Cos } \theta$$

$$\frac{\text{Perpendicular}}{\text{Base}} = \text{tan } \theta$$

Misconceptions/Common errors

i) Student some times take Sin θ, Cos θ, and tan θ seperately as Sin and θ, cos and θ, tan and θ while they are not separable

Sin θ means sine of θ and not the product of sin and θ. So is the case with other trigonometric functions.

ii) In simplification of trigonometrical functions

$\frac{\text{Sin}\,\theta}{\text{Cos}\,\theta}$ and $\frac{\text{Sin}\,2\theta}{\text{Cos}\,2\theta}$

θ .. 2 θ can not be cancelled out, here sin θ, sin θ, sin 2 θ, cos θ, cos 2 θ, are independent term in themselves.

iii) Student should also be made clar that all trigonometric functions in first quadrant have positive value and in other quadrants different functions have positive or negative sign according to their nature.

iv) Values of Trigonometric functions can be derived from Tables of Trigonometric functions, given at the end of the text book.

v) Some times students writes

$\sin (A-B) = \sin A - \sin B$

$\sin (90-30) = \sin 60 = +\frac{\sqrt{3}}{2}$

$\sin 90 - \sin 30 = 1 - \frac{1}{2} = \frac{1}{2}$

i.e. $\sin (90-30) \neq \sin 90 - \sin 30$

so is the case with $\cos (90-30) \neq \cos 90^\circ - \cos 30^\circ$

<u>Additional Examples:</u>

1. If $A + B + C = \pi$

Show that $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C$

Sol. <u>L.H.S</u> $= \sin^2 A + \sin^2 B - \sin^2 C$

$= \sin^2 A - \sin^2 C + \sin^2 B$

$= \sin (A+C) . \sin (A-C) + \sin^2 B$

$= \sin B . \sin (A-C) + \sin^2 B$

$\because \sin (A+C) = \sin B$

$= \sin B \left[\sin (A-C) + \sin B \right]$

$= \sin B \left[\sin (A-C) + \sin (A+C) \right]$

$= \sin B . 2 \sin A \cos C$

$= 2 \sin A \sin B \cos C$

$=$ R.H.S.

Alternate method -

The same problem can be solved by using the formula

$\mathrm{Sin}^2 A = \frac{1-\cos 2A}{2}$ etc. then L.H.S. will become

$$= \frac{1-\cos 2A}{2} + \frac{1-\cos 2B}{2} - \frac{1-\cos 2C}{2}$$

$$= \frac{1}{2}\left[1 - \mathrm{Cos}\ 2A - \mathrm{Cos}2B + \mathrm{Cos}2C\right]$$

$$= \frac{1}{2}\left[1 - \mathrm{Cos}2A - (\mathrm{Cos}2B - \mathrm{Cos}\ 2C)\right]$$

$$= \frac{1}{3}\left[1 - \mathrm{Cos}2A + 2\mathrm{Sin}\ (B+C)\ \mathrm{Sin}\ (B-C)\right]$$

$$= \frac{1}{2}\left[2\ \mathrm{Sin}^2 A + 2\mathrm{Sin}\ A\ \mathrm{Sin}\ (B-C)\right]$$

$$= \frac{1}{2}\ 2\mathrm{Sin}\ A\left[\mathrm{Sin}\ A + \mathrm{Sin}\ (B-C)\right]$$

$$= \mathrm{Sin}\ A\left[\mathrm{Sin}\ (B+C) + \mathrm{Sin}(B-C)\right]$$

$$= 2\ \mathrm{Sin}\ A\ \mathrm{Sin}\ B\ .\ \mathrm{Sin}\ C$$

2. If $A + B + C = \pi$ Prove that

$$\cos^2 \frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2+2\sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2} \qquad (1)$$

$\because A + B + C = \pi$

$\therefore \frac{A}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{C}{2}$

i.e. $\cos \frac{A + B}{2} = \cos\left(\frac{\pi}{2} - \frac{C}{2}\right) = \sin \frac{C}{2}$ etc.

Soln. Thus L.H.S. of the (1) is.

$$= \frac{1}{2}\left[2 \cos^2\frac{A}{2} + 2\cos^2\frac{B}{2} + 2 \cos^2\frac{C}{2}\right]$$

$$= \frac{1}{2}\left[1+ \cos A + 1 + \cos B + 1 + \cos C\right]$$

$$= \frac{1}{2}\left[4 + 2 \cos \frac{A + B}{2} \cdot \cos \frac{A - B}{2} - (1-\cos C)\right]$$

$$= \frac{1}{2}\left[4 + 2 \sin \frac{C}{2} \cdot \cos \frac{A-B}{2} - 2\sin^2\frac{C}{2}\right]$$

$$= \frac{2}{2}\left[2 + \sin \frac{C}{2} \left(\cos \frac{A-B}{2}\right) - \cos \left(\frac{A+B}{2}\right)\right]$$

$$\because \sin \frac{C}{2} = \cos \frac{(A+B)}{2}$$

$$= 2 + \sin \frac{C}{2} \left[\cos \frac{A - B}{2} - \cos \frac{A + B}{2}\right]$$

$$= 2 + 2 \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2}.$$

Alternate method:

The above problem can also be solved by using the formula:

L.H.S. $= \cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2}$

$= 1 - \sin^2\frac{A}{2} + 1 - \sin^2\frac{B}{2} + \cos^2\frac{C}{2}$

$= 2 - \sin^2\frac{A}{2} - (\sin^2\frac{B}{2} - \cos^2\frac{C}{2})$

$= 2 - \sin^2\frac{A}{2} + \cos\frac{B+C}{2} \quad \cos\frac{B-C}{2}$

$= 2 - \sin^2\frac{A}{2} + \sin\frac{A}{2} \cdot \cos\frac{B-C}{2}$

$= 2 + \sin\frac{A}{2}\left[\cos\frac{B-C}{2} - \cos\frac{(B+C)}{2}\right]$

$= 2 + 2\sin\frac{A}{2}\ \sin\frac{B}{2}\ \sin\frac{C}{2}$

5. <u>Chapter Test</u>

(a) Oral Test

1. Express following radian measurements in degrees

(a) $\frac{2}{3}\pi$ radian

(b) $\frac{1}{3}\pi$ radian

(c) $\frac{3}{2\pi}$ radian

(d) $\frac{3}{\pi}$ radian

2. If $\sin A = \frac{1}{2}$ & $\sin B = \frac{1}{\sqrt{2}}$ then what will be value of

$\sin(A+B)$? Ans. $\left[\frac{\sqrt{3}+1}{2\sqrt{2}}\right]$

3. If the difference between two acute angles of a right triangle is $\frac{\pi}{9}$, then calculate the angles in degrees. Ans. (55°, 35°)

4. What will be the expression formula for

   (a) Cos C - Cos D ?

   (b) 2 Cos A Sin B

   (c) Sin 3 A

5. What will be value of θ in the equations ?

   (a) $\sin\theta = \frac{1}{2}$

   (b) $\cos 2\theta = \frac{\sqrt{3}}{2}$

   (c) $\tan\theta = \sqrt{3}$

5. Chapter Test : Written Test

Q1. Fill up the blanks in the following:

   (a) The degree measure of an angle is 810°, its radian measure is _______________.

   (b) The value of $\frac{2\pi}{3}$ radians in degree will be_______

   (c) The angle between the minute hand of a clock and the hour hand when the time is 3-40.

Q2. If $\sin\alpha = \frac{3}{5}$ find $\cos\alpha$ and $\tan\alpha$

Q3. If $\tan\theta = \frac{5}{12}$ in 3rd quadrant find the value of other five functions.

Q4. Prove that

$$(\sec\theta - \cos\theta)\ (\operatorname{cosec}\theta - \sin\theta) = \frac{1}{\tan\theta + \cot\theta}$$

Q5. Prove that

(a) $\sin A (1 + \tan A) + \cos A (1 + \cot A) = \sec A + \operatorname{cosec} A$

(b) $\sin A + \sin (120 + A) + \sin (240 + A) = 0$

Q6. Show that

(a) $\sec^6 \alpha - \tan^6 \alpha = 1 + 3\tan^2 \alpha + 3 \tan^4 \alpha$

(b) $\sqrt{\frac{1+\cos A}{1-\cos A}} = \operatorname{cosec} A + \cot A$

Q7. If $\sin \alpha = \frac{5}{13}$ and $\cos \beta = \frac{3}{5}$ find value of

(a) $\sin (\alpha + \beta)$

(b) $\cos (\alpha - \beta)$

Q8. Prove that

(a) $\sin \theta + \cos \theta = \sqrt{2} \sin (45^\circ + \theta)$

(b) $\cot A + \cot B = \frac{\sin (A+B)}{\sin A \sin B}$

(c) $\tan \frac{\pi}{6} + \tan \frac{\pi}{12} + \tan \frac{\pi}{6} . \tan \frac{\pi}{12} = 1$

Q9. If $\sin (\theta + \alpha) = n \sin (\theta - \alpha)$,

then prove that $\cot \theta = \frac{n-1}{n+1} \cot \alpha$

Q10. Find the value of following (By use of table)

(a) $\sin 34^\circ 22'$

(b) $\cos 20^\circ 10'$

(c) $\tan 54^\circ 30'$

Q11. Trane the following graph

$y = \cos x$, $y = 3 \sin 2x$

Q12. In ΔABC, prove that

(a) $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} - \cos^2\frac{C}{2} = 2\cos\frac{A}{2} \cos\frac{B}{2} \sin \frac{C}{2}$

(b) $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$

Q13. If $A+B+C = \pi$, show that

(a) $\cos 2A + \cos 2B - \cos 2C = 1 - 4 \sin A \sin B \cos C$,

(b) $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \sin B \cos C$.

Q14. Solve the equations

(a) $\sqrt{3} \cos \theta + \sin \theta = \sqrt{2}$

(b) $\cos 3\theta + \cos \theta - 2 \cos 2\theta = 0$

Q15. Solve the equation

$2 \tan \theta - \cot \theta = -1$

Key to Chapter Test (Written Test)

1 (a) $4\pi + \frac{\pi}{2}$ or $9 \frac{\pi}{2}$

(b) 120^{o}

(c) 150^{o}

2. $\cos\alpha = \frac{4}{5}$

$\tan\alpha = \frac{3}{4}$

3. $\sin\theta = \frac{5}{13}$

$\cos\theta = \frac{12}{13}$

$\cot\theta = \frac{12}{5}$

$\sec\theta = \frac{13}{12}$

$\operatorname{cosec}\theta = \frac{13}{5}$

7. (a) $\frac{63}{65}$

(b) $\frac{56}{65}$

10. (a) .5645

(b) .9387

(c) 1.4020

14. (a) $\theta = 2n\pi + \frac{5\pi}{12}$ or $2n\pi - \frac{\pi}{12}$

(b) $\theta = (2n+1)\frac{\pi}{4}$ or $2n\pi$, where $n, m \in I$

15. $\theta = n\pi + (-1)^{n} \frac{\pi}{2}$

$= m\pi + (-1)^{m} \frac{\pi}{6}$, $m, n \in I$

Additional Problems: Unit . I

1. If $\sin\theta = \frac{21}{29}$ and θ lies in I quadrant, then find the value of $\sec\theta \pm \tan\theta$ [(Ans $2\frac{1}{2}$)]

2. If $\tan A = \frac{2mn}{m^2-n^2}$ find cos A and Cosec A. A being acute. [(Ans: $\frac{m^2-n^2}{m^2+n^2}$)]

3. If $\sec A = \frac{13}{5}$ find $\frac{2\sin A + 3\cos A}{4\sin A - 9\cos A}$ A being acute. [(Ans: 13)]

4. If $\cot\theta = \sqrt{-1}$ evaluate $\frac{\operatorname{cosec}^2\theta + \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta}$ Ans$(\frac{4}{3})$

5.

Unit - II

Prove the following identities

1. $\sin^3 A - \cos^3 A = (\sin A - \cos A)(1+\sin A\cos A)$

2. $\sin^4\theta + \cos^4\theta = 2\sin^4\theta - 2\sin^2\theta + 1$

3. $\cos^6\theta + \sin^6\theta = 1-3\sin^2\theta\cos^2\theta$

4. $\frac{\sec\theta}{\cot\theta + \tan\theta} = \sin\theta$

5. $\frac{1}{\sec\theta + \tan\theta} = \sec\theta - \tan\theta$

6. $(\tan\theta - \cot\theta)(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta) = \sin^2\theta - \cos$

7. $(\sin A - \operatorname{cosec} A)^2 + (\cos A - \sec A)^2 = \cot^2 A + \tan^2 A - 1$

8. $\dfrac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta} = 1 - 2\sec\theta\tan\theta + 2\tan^2\theta$

9. $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\tan\theta}{\cot\theta} = \dfrac{\sin^2\theta\,(1+\cot\theta)}{\cos^2\theta}$

10. $\sqrt{\operatorname{Sec}^2 A + \operatorname{Cosec}^2 A} = \tan A + \operatorname{Cot} A.$

11. $\sqrt{\dfrac{1+\tan^2 A}{1+\cot^2 A}} = \dfrac{\tan A - 1}{1 - \cot A}$

12. $\dfrac{\tan A + \operatorname{Cot} B}{\cot A - \tan B} = \dfrac{\tan A}{\tan B}$

13. $\tan^2 A - \tan^2 B = \dfrac{\operatorname{Sin}^2 A - \operatorname{Sin}^2 B}{\operatorname{Cos}^2 A \times \operatorname{Cos}^2 B}$

14. $(\operatorname{Sin} x + \operatorname{cosec} x)^2 + (\cot x + \sec x)^2 = 7 + \tan^2 x \cot^2 x$

15. $(1+\cot A - \operatorname{Cosec} A)(1+\tan A + \sec A) = 2,$

16. $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) = 13.$

17. $\dfrac{\tan x + \tan y}{\cot x + \cot y} = \tan x \tan y$

18. $\cot^2 y\,\dfrac{\sec y - 1}{1+\sin y} + \sec^2 y\,\dfrac{\sin y - 1}{1+\sec y} = 0$

19. $\dfrac{1+\sin A - \operatorname{Cos} A}{1+\operatorname{Sin} A + \cos A} + \dfrac{1+\operatorname{Sin} A + \operatorname{Cos} A}{1+\operatorname{Sin} A - \operatorname{Cos} A} = 2\operatorname{cosec} A$

20. $(\operatorname{Sec}\theta + \operatorname{cosec}\theta)^2 = (1+\tan\theta)^2 + (1+\cot\theta)^2$

21. If $\tan\theta + \operatorname{Sec}\theta = a$, show that $2\tan\theta = a - \frac{1}{a}$, $2\sec\theta = a + \frac{1}{a}$,

Hence show that $\sin\theta = \dfrac{a^2 - 1}{a^2 + 1}$

22. Eliminate ϕ from the following equations

$x = a\cos\phi + b\sin\phi$

$y = a\sin\phi - b\cos\phi$ (Ans $x^2+y^2=a^2+b^2$)

23. If $\tan A + \sin A = m$, $\tan A - \sin A = n$, prove that $m^2-n^2 = 4\sqrt{mn}$

24. If $x = a\cos^3\theta$, $y = b\sin^3\theta$, show that $(\frac{x}{a})^{\frac{2}{3}} + (\frac{y}{b})^{\frac{2}{3}} = 1$.

Unit - III

1. Find the value of $\sin^2 30^\circ + \sin^2 45^\circ + \sin^2 60^\circ$ (Ans: $1\frac{1}{2}$)

2. Find the value of $3\tan^2 30^\circ + \frac{1}{4}\sec^2 60^\circ + 5\cot^2 45^\circ - \frac{2}{3}\sin^2 60^\circ$

(Ans: 6)

3. Prove that $\cos 24^\circ + \cos 55^\circ + \cos 125^\circ + \cos 204^\circ + \cos 300^\circ = \frac{1}{2}$

4. Find the value of $\sin 780^\circ \cos 30^\circ + \cos 120^\circ \sin 390^\circ$

(Ans $\frac{1}{2}$)

5. Simplify: $\dfrac{\sin(180^\circ+A)\ \cos(90^\circ-A)\ \tan(270^\circ-A)}{\sec(540^\circ-A)\ \cos(360^\circ+A)\operatorname{cosec}(270^\circ+A)}$

(Ans : $\sin A \cos^2 A$)

Unit - IV

1. If $\sin A = \frac{3}{5}$, $\cos B = \frac{12}{13}$ find $\sin(A-B)$ and $\cos(A+B)$

(Ans $\frac{16}{65}$, $\frac{33}{65}$)

2. Show that $\sin A + \sin(120^\circ+A) + \sin(240^\circ+A) = 0$.

3. Find $\tan 75^\circ$, show that $\tan 75^\circ + \cot 75^\circ = 4$

4. Prove that $\frac{\text{Sin } 3A}{\text{Sin } A} + \frac{\text{Cos } 3A}{\text{Cos } A} = 4 \text{ Cos } 2A$

5. Show that $\tan 5x - \tan 3x - \tan 2x = \tan 5x \tan 3x \tan 2x$.

6. Show that $4(\text{Cos}^3 20^\circ + \text{Cos}^3 40^\circ) = 3(\cos 20^\circ + \cos 40^\circ)$

7. If $\sin x + \sin y = a$, $\text{Cos } x + \cos y = b$,

find the value of $\tan^2\left(\frac{x-y}{2}\right)$

(Ans: $\frac{4-a^2-b^2}{a^2+b^2}$)

<u>List of Reference Book</u>:

1. Plane Trigonometry - By S.L. Loney.

2. Trigonometry - By S. Narayanan and T.K. Manic a vachagon Pillai

3. Mathematical Handbook - By M. V.Y.Godsky
Elementary Mathematics
[illegible] Mir Publishers, Moscow

4. Plane Trigonometry By S.P. Nigam
B.S. Tyagi

5. A new Book of Mathematics - By C.S. Sarana
R.G.Gupta
P.K.Garg
Arya Book Depot,
Karol Bagh, Delhi.

CHAPTER 14

SOLUTION OF TRIANGLES

I. Introduction

A triangle has three angles and three sides, and they are termed as parts of a triangle. If we have the measures of three parts then other three parts of the triangle can be easily derived. However, knowing three angles of triangle is just equivalent to knowing two angles and in such situation we have to know one more side of the triangle so that we can construct the triangle or solve the triangle.

This chapter deals with the solution of triangles or finding the unknown parts of the triangle, when the measures of three independent parts are given. The knowledge of solution of triangle is useful in finding the heights and distances. However the knowledge of solving a triangle is used in navigation, astronomy and in other sciences.

2. Content Analysis

In this section the number of each subsection is in accordance with the textbook.

14.1 Some Basic Formulae.

1. Parts of a Triangle

(a) Three sides and three angles of a triangle are termed as parts of the triangle.

(b) Knowledge of two angles of a triangle is sufficient to find the third angle.

(c) Any triangle can be solved if three independent parts of the triangle are known to us. Knowledge of three angles is not taken as three independent parts of the triangle.

(d) With the help of three known parts of the triangle, the other parts of the triangle can be calculated. The process is known as solution of the triangle.

2. When two sides and one angle or two angles and one side of the triangle are given, we can determine the third side and the other two angles or the other two sides and the third angle of the triangle with the help of the relation

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

for a Δ ABC.

3. When two sides and an angle is given, we use cosine formula known as law of cosine to find the third side and the other angles.

The formulae are given below :

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = c^2 + a^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

<u>Half Angles</u> Using the formula $\cos A = \frac{b^2 + c^2 - a^2}{2\,ab}$

in $2 \sin^2 \frac{A}{2} = 1 - \cos A$

We get $2 \sin^2 \frac{A}{2} = 1 - \frac{b^2 + c^2 - a^2}{2\,bc}$

$$= \frac{a^2 - (b - c)^2}{2\,bc}$$

$$= \frac{(a + b - c)\,(a - b + c)}{2\,bc}$$

In article 14.1, the formulae based on $a + b + c = 2s$ are derived for $\sin\frac{A}{2}$, $\sin\frac{B}{2}$, $\sin\frac{C}{2}$ and $\cos\frac{A}{2}$, $\cos\frac{B}{2}$, $\cos\frac{C}{2}$.

With the help of the figures 14.1 and 14.2 as given in the textbook, the area of a triangle ABC is derived as

$$\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B$$

or $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \dfrac{a+b+c}{2}$

This formula is known as Hero's formula.

14.2 Some more Formulae

With the help of the Fig. 14.1 and 14.2 of the textbook, relation between sides and angles are established as follows :

1. $a = b \cos C + c \cos B$
2. $b = c \cos A + a \cos C$
3. $c = a \cos B + b \cos A$
4. $\tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c} \cot\dfrac{A}{2}$
5. $\tan\dfrac{C-A}{2} = \dfrac{c-a}{c+a} \cot\dfrac{B}{2}$
6. $\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b} \cot\dfrac{C}{2}$

Half angle formula

With the help of Fig. 14.3 in the textbook, we establish relations among the radius r of the incircle, radius R of the circumcircle and the area Δ of a triangle. These relations are given

$$r = \frac{\text{Area of the } \Delta}{s} \; ; \; \frac{r}{s-a} = \tan\frac{A}{2}$$

$$\frac{r}{s-b} = \tan\frac{B}{2}, \; \frac{r}{s-c} = \tan\frac{C}{2} \text{ and}$$

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}.$$

The solved examples of 14.3 to 14.5 are helpful in solving questions of exercise 14.3.

14.3 Right Triangles

In such triangles if we know one side and one more angle then we can solve the triangle with the help of formulae

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

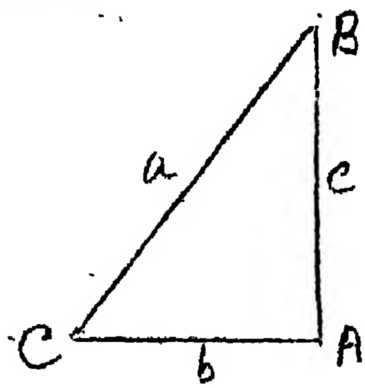


Fig. 14.1

where A is 90°, if $\angle B$ and b are given

then $\frac{a}{\sin 90^\circ} = \frac{b}{\sin B}$.

$$\therefore \; a = \frac{b}{\sin B},$$

and $c = \sqrt{a^2 - b^2} = \sqrt{\frac{b^2(1-\sin^2 B)}{\sin^2 B}}$

$c = b \cot B.$

Examples 14.6 and 14.7 will be helpful in solving exercise 14.4

14.4 Oblique Triangles

In solving problems related to oblique Δ^s : four different situations are possible as given in the text book.

14.5 Heights and Distances

The most effective use of trigonometric functions is in solving problems on heights and distances.

In solving such problems the teacher should distinguish between angle of elevation and angle of depression, and how they are measured and represented in the figure. In solving problems following steps are being followed :

1. Figure should be drawn according to the data or facts given in the problem.

2. Establish the relation between the angle of elevation, angle of depression or distances given.

3. With the help of trigonometric functions and known facts of triangle, the other parts of the triangles are found out.

Angles of Elevation and Angles of Depression

If h is the height of the tower, and P is a point at a distance x from the bottom of the tower, $\angle$ BPA $= \alpha$ is the angle of elevation of the top of the tower from the point P, and $\angle$ PAN $= \theta$ is the angle of depression, from the top of the tower.

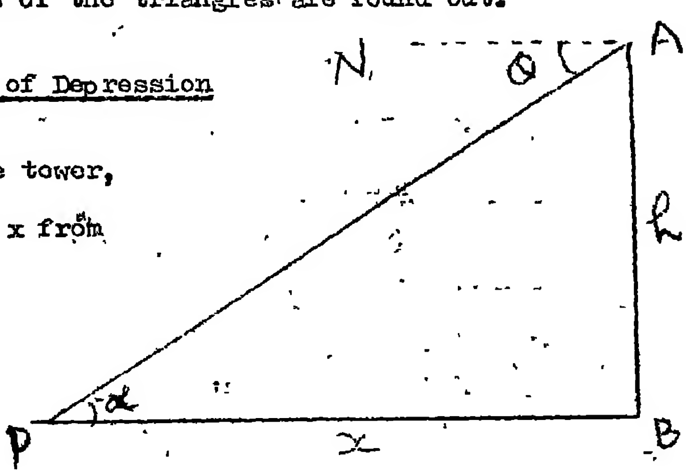


Fig. 14.2

To Find Height of the tower :

By the property of right angles, we have

$$\frac{h}{x} = \tan\alpha.$$

Out of three unknown quantities the knowledge of two measures will be sufficient for calculating the third.

$\therefore$ $h = x\tan\alpha$ if x and α are known.

<u>Direction</u>

In tracing the figure we must give proper attention to directions East, West, South, North or N.E., S.E., N.W. and S.W. This will help the child in proper illustration of the problem.

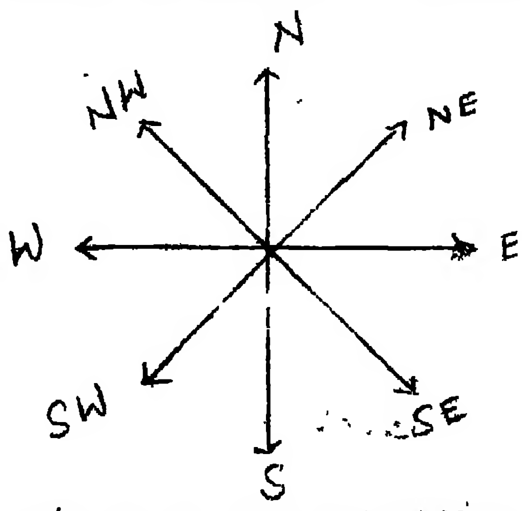


Fig. 14.3

<u>Solution/Hints for Difficult Problems</u>

<u>Exercise 14.1</u>

<u>Q.No.1.</u> Here $a = 18$, $b = 24$, $c = 30$ with the help of Hero's formulae.

$$\text{Area of the } \triangle = \sqrt{(36)\ (18)\ (12)\ (6)}$$

$$= 216.$$

(i) Then $\sin A = \dfrac{2 \text{ Area of } \Delta}{bc} = \dfrac{2 \times 216}{24.30}$

$$= \frac{3}{5} \text{ or } 0.6$$

(ii) To find the value for tan A, we first find the value of Cos A from the formula

$\cos A = \dfrac{b^2 + c^2 - a^2}{2\,bc}$ which is $\dfrac{4}{5}$

$\therefore \tan A = \dfrac{3}{5} \div \dfrac{4}{5} = \dfrac{3}{4}$

Similarly the value for other function can be found out.

(iii) The area of triangle is already calculated above with the help of Hero's formula and it is 216 squ. units.

(iv) The value of $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ can be calculated with the help of formulae

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{s\,(s-a)}}, \quad \tan\frac{B}{2} = \sqrt{\frac{(s-c)\,(s-a)}{s\,(s-b)}}$$

and $\tan\dfrac{C}{2} = \sqrt{\dfrac{(s-a)\,(s-b)}{s(s-c)}}$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{s\,(s-a)}}$$

$$= \sqrt{\frac{12.6}{36.18}}$$

$$= \frac{1}{3}$$

Here $a = 18$, $b = 24$, $c = 30$

$s = \dfrac{a+b+c}{2} = 36$

$s - a = 18$

$s - b = 12$

$s - c = 6.$

Here we reject $\tan\frac{A}{2} = -\frac{1}{3}$ as $\frac{A}{2}$ can **never** exceed $\frac{\pi}{2}$.

Similarly, with the help of proper formulae the value of $\tan\frac{B}{2}$ and $\tan\frac{C}{2}$ can be calculated.

Questions of Exercise 14.5 are solved with the help of Example 14.8.

<u>Exercise 14.6</u>

<u>Question No.4</u>

Solution : $a = 40$, $c = 40\sqrt{3}$ and $B = 30^\circ$ solve the triangle.

Since $B = 30^\circ$, $A + C = 150^\circ$ .... (1)

Also using formula $\tan\frac{C-A}{2} = \frac{c-a}{c+a} \cot\frac{B}{2}$

We get $\tan\frac{C-A}{2} = \frac{40(\sqrt{3}-1)}{40(\sqrt{3}+1)}$, $\cot 15^\circ$

$= 1 \qquad \because \cot 15^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

$= \tan 45^\circ$

$\therefore \frac{C-A}{2} = 45^\circ$ or $C - A = 90^\circ$ ...... (2)

With the help of (1) and (2) we get

$A = 30^\circ$ and $C = 120^\circ$

For getting the other side b, we use the formula $\frac{\sin B}{b} = \frac{\sin A}{a}$ $\therefore b = \frac{a \sin B}{\sin A}$

$= \frac{a \sin 30^\circ}{\sin 30^\circ}$

$b = a = 40$

Hence we can determine the parts of the triangle

$A = 30^\circ$ $B = 30^\circ$, $C = 120^\circ$

$a = 40$ $b = 40$ $c = 40\sqrt{3}$.

Exercise 14.9

Q.No.1 With the help of the figure

$\frac{AB}{AC} = \tan 58^\circ$

$\therefore AB = AC \tan 58^\circ$

$= 9.6 \times 1.6$

$= 15.36$ meter.

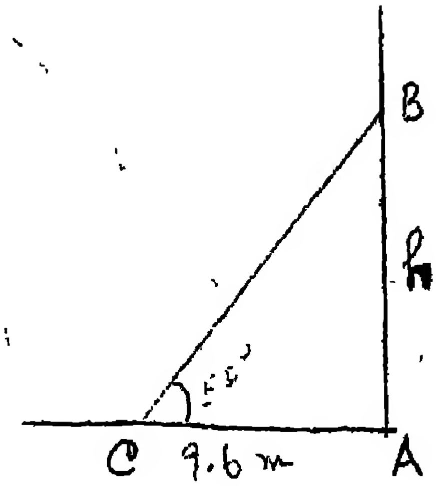


Fig. 14.4

Q.No.4 With the help of the figure

$OA = h \cot\alpha$

$OB = h \cot\beta$

$OA - OB = h(\cot\alpha - \cot\beta)$

$240 = h\left(\frac{12}{5} - \frac{4}{3}\right)$

$\therefore h = 240 \times \frac{15}{16} = 225$ meter.

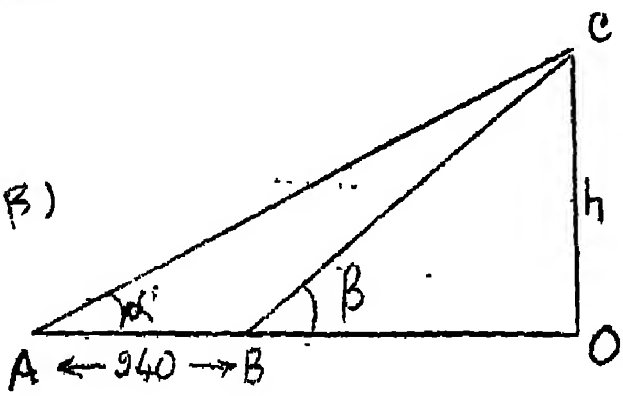


Fig. 14.5

Q.No.8

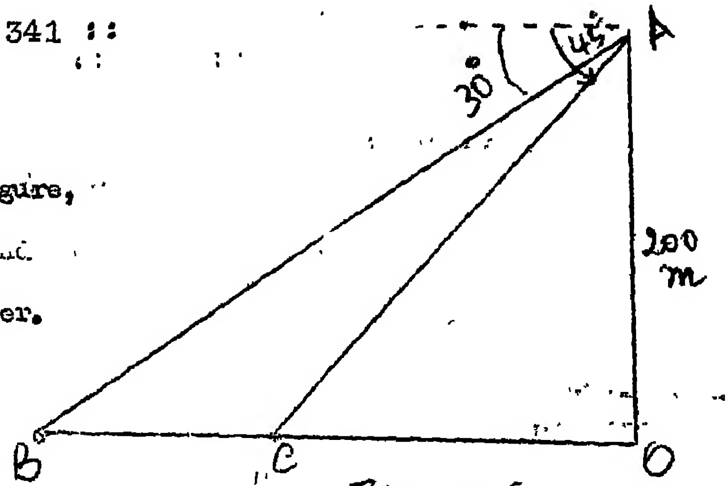


Fig. 14.6

With the help of the figure,

B and C are two ships and

A is the top of the tower.

$OB = 200 \text{ Cot } 45^\circ$

$OC = 200 \text{ Cot } 30^\circ$

Which gives $OB - OC = 200 \left[\text{Cot } 30^\circ - \text{Cot } 45^\circ\right]$

$200 \left[\sqrt{3} - 1\right]$

Distance between the two ships is

$= 200 \times \left[1.732 - 1\right]$

$= 200 \times .732$

$= 146.4$ meter.

3. Learning Outcomes:

(a) Essential Learning Outcomes For All.

Learner should be able to :

1. define and recognise parts of a triangle.
2. Use sine and cosine formulae for the solution of triangles.
3. Use Half Angle formulas and to calculate area of triangle with the help of Hero's formula.
4. Find radii of inscribed and circumscribed circles.
5. Calculate the sides of a triangle in terms of radii of circumscribe circle and angles of the triangle.
6. Solve right triangles and oblique triangles.
7. Use the tables of trigonometric functions.

8. Recognise ambiguous cases in solution of triangles and find out parts of such triangles with the help of tables for trigonometric functions.

9. Solve problems based on heights and distances with the help of trigonometric functions.

(b) Learning Outcomes For The Higher Group :

1. Learner should be able to sketch the difficult problems and solve them.

2. Learner should be able to solve the ambiguous triangle and find out both solutions for it.

4. Teaching Strategies:

Motivation

In order to arouse the interest of the child, motivational activities of the following sort are suggested :

(i) Draw a triangle as in the figure and ask the students to find

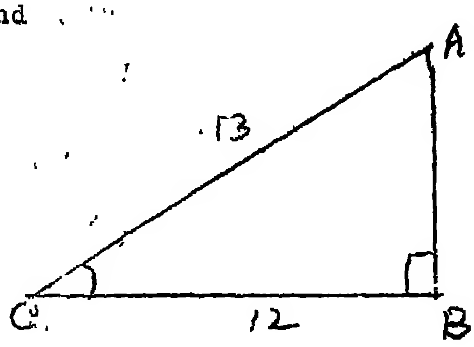


Fig. 14.7

(a) The sum of other two angles in Δ ABC

(b) The measure of the side AB

(c) The relation which helps in finding these measures.

(d) When two sides and one angle is given in a right angled triangle, which formulae are useful in deriving other results ?

Fig. 14.7

(e) When measures of three sides are given, how can you calculate the area of the triangle ?

(ii) How the measures of sides of a triangle are helpful in calculating its angles ?

(iii) How radii of incircle and circumcircle of a triangle are used in calculating area of the triangle and other parts of the triangle.

(iv) Teacher will display the different types of triangles and ask the students to differentiate right triangle, oblique triangle and ambiguous triangle.

(v) Teacher may explain the situations in which height of a mountain, tower or window in a wall, distance accross the river or distance between two ships sailing on the sea is to be determined.

(a) How the situations are sketched ?

(b) How the distances or heights are being calculated.

(c) Redescribe the situation using mathematical knowledge and then we check to see if we have the solution.

(d) The mathematical redescription usually involves the use of one or more trigonometric ratios chosen in such a way that it relates the known and unknown parts of a right triangle.

<u>Misconceptions and Common Errors</u>

(i) Some times students do not recall the proper formula.

(ii) In using the Hero's formula for area of a triangle or formulae for half angles as $\sin\frac{A}{2}$, $\cos\frac{A}{2}$, $\tan\frac{A}{2}$ student usually use s as sum of three sides of the triangle instead of half of the sum of the three sides of a triangle.

(iii) Writing relationship between radii of incircle and circumcircle, area of triangle and angles of a triangle, students do not use proper formulae at proper place.

(iv) Sometimes student commit mistake in denoting angle of elevation and angle of depression.

(v) In calculations involving logarithmic tables, students usually commit mistake in using log Sin θ, Log Cos θ, L $\sin\theta = 10 + \log\sin\theta$ and L $\cos\theta = 10 + \log\cos\theta$

<u>Additional Exercises:</u>

Properties of triangle :

1. In any $\triangle$ ABC, prove that

$$2\left(a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2}\right) = c + a - b$$

2. Prove that, in any $\triangle$ ABC

$$(a+b+c)\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right) = 2c\cot\frac{C}{2}$$

3. If $\cot\frac{C}{2} = \frac{a+b}{c}$, prove that the triangle ABC is right angled.

4. In a triangle measurement of the sides are 6, 10, and 14 respectively. Show that the triangle is obtuse and that the obtuse angle is 120°.
(Hint : Use $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$ and solve)

5. In any $\triangle$ ABC prove that

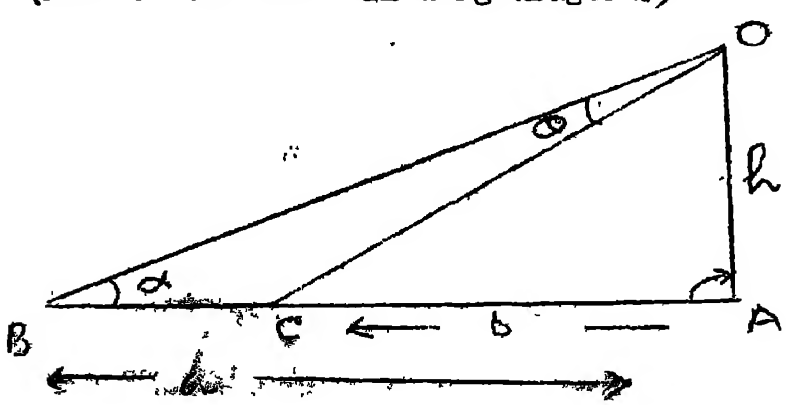
$$\frac{\cos A}{\cos B} = \frac{b - a \cos C}{a - b \cos C}$$

6. Solve the following triangles, when —

(i)	$a = 32,$	$b = 40,$	$c = 66$
(ii)	$b = 130$,	$c = 72,$	$A = 42^\circ$
(iii)	$a = 152$,	$A = 80^\circ$	$B = 53^\circ$
(iv)	$b = 105$,	$C = 150$	$B = 52^\circ\ 30'$
(v)	$a = 38.8,$	$b = 42.9$	$C = 38^\circ 16'$

7. The angles of elevation of the top of a vertical tower from two points distant a, and b from the base and in the same line with it are complementary. Prove that the height of the tower is $\sqrt{ab}$.

If θ is the angle subtended at the top of the tower by the line joining these points, then prove that $\sin \theta = \dfrac{a-b}{a+b}$.
(Hint : Use the following diagram.)



8. A vertical tower stands on a horizontal plane and is surmounted by a vertical flag staff of height h. At a point on the plane, the angle of elevation of the bottom of the flagstaff is α and that of the top of the flag staff is β. Prove that the height of the tower is $\dfrac{h\tan\alpha}{\tan\beta-\tan\alpha}$

9. From the top of a cliff, 200 m high, the angles of depression of the top and bottom of a tower are observed to be 30° and 60°. Find the height of the tower. (Ans: 133.33 m)

10. A man on the top of a rock rising on a seashore observes a boat coming towards it. If it takes 10 minutes for the angle of depression to change from 30° to 60°, how soon will the boat reach the shore ?

(Ans: In 5 minutes)

<u>Hint</u>:

Let h be the height of the rock. Y the distance the boat travelled in 10 minutes and x the distance it has to travel to reach the shore.

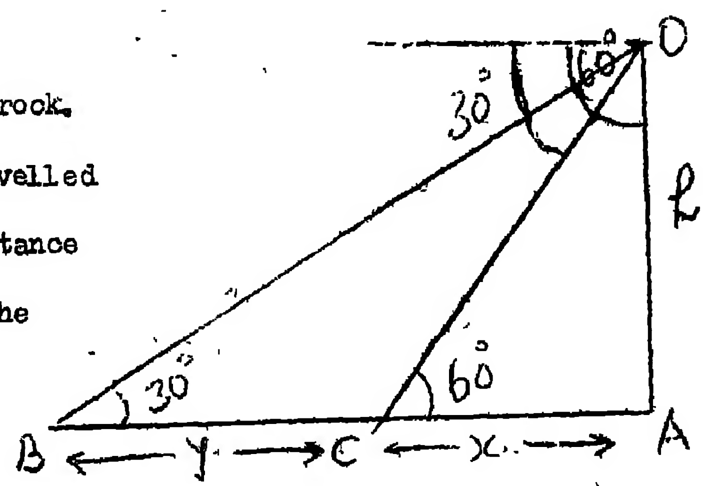


Fig. 14.9

$\tan 60^\circ = \frac{h}{x}$ $\tan 30^\circ = \frac{h}{x+y}$

$\therefore$ $h = x\sqrt{3}.$ $h = \frac{x+y}{\sqrt{3}}$

by solving $x\sqrt{3} = \frac{x+y}{\sqrt{3}}$ we get

$x = \frac{1}{2}y$ i.e. the boat will reach the shore in 5 minutes.

11. If the angle of elevation of a cloud from a point h metre above a lake be β and the angle of depression of its reflection in the lake be α, then find the height of the cloud.

(Ans: height of the cloud $x = \frac{h \sin(\alpha+\beta)}{\sin(\alpha-\beta)}$ meter.

5. **Chapter Test :**

Oral Test

1. Angle of depression of a point on earth from the top of a tower whose height is 60 m is 30°. Find the distance of the point from the foot of the tower.

(Ans $(60\sqrt{3})$ m.

2. Length of the shadow of a pole is $\frac{1}{\sqrt{3}}$ time the height of the pole. What is the angle of elevation of the sun?

$(\alpha = 30^\circ)$

3. Find the angle of elevation, when the length of the shadow of a pole is $\sqrt{3}$ times the height of the pole.

4. What is the value of

(a) $\sin(A + B)$ (c) $\sin(A - B)$

(b) $\cos(A + B)$ (d) $\cos(A - B)$

5. If $\cos B = \frac{3}{\sqrt{10}}$ what will be the value of tan 13 ? Prove that $A + B = 45^\circ$

6. Give the Cosine formulae when sides a, b, & c of a triangle are given.

7. What types of triangles are formed when two sides and angle opposite to one of them is given ?

8. What is ambiguous case in solution of a triangle ?

(a) <u>Written Test</u>

1. Find the area of a triangle, when the measures of its three sides are $a = 4$, $b = 5$, $c = 3$

(Ans: 6 Sq. units)

2. In a triangle ABC, $a = \sqrt{3}$, $\angle A = 60^\circ$ and $\angle C = 45^\circ$; Find C.

(Ans: $c = \sqrt{2}$)

3. Solve the triangle when $b = 3$, $c = 3\sqrt{3}$ and $\angle B = 30^\circ$

Ans: $\angle C = 60^\circ$
$\angle A = 90^\circ$
$a = 6$

10. Angles of depression of the top and bottom of a tower from the top of a light house are 45° and 60° respectively. If the height of the light house is 600 metres find out the height of the tower.

Ans: $200\sqrt{3}(\sqrt{3}-1)$ m.

Reference Books

(As mentioned in the Chapter 13).

1. Plane Trigonometry — By S.L. Loney
2. Trigonometry — By N. Narayanan and T.K. Manicavachgan Pillai
3. Mathematical Handbook Elementary Mathematics — By M. Vygodsky, Mir Publishers, Moscow
4. A Text Book on Plane Trigonometry — By S.P. Nigam, B.S. Tyagi
5. A New Book of Mathematics — By C.S. Sarma, R.G. Gupta, P.K. Garg, Arya Book Depot, Karol Bagh, New Delhi.

CHAPTER-15

INVERSE TRIGONOMETRIC FUNCTIONS

1. INTRODUCTION:

Students are already familiar with the definition of inverse function of a given function and the condition for the existence of inverse function of a given function. This idea will be extended to trigonometric functions in this chapter.

The concept of inverse functions has a number of applications in Trigonometry, Number Theory, Real and Complex Analysis, Functional Analysis, Abstract Algebra, Topology and several other branches of Mathematics. We have implicitly used the concept of inverse functions in the Chapter 7 of the text-book. The concept of inverse trigonometric functions will be used vividly in Differential and Integral Calculus in the subsequent classes.

2. CONTENT ANALYSIS:

In this section, the number of each subsection is in accordance with that in the text book.

15.1 The Inverse of a Function

The students have an idea regarding 'one-one' and 'onto' function. It may be made clear to the students that inverse of a function exists iff it is one-one as well as onto.

The equation $\sin\theta = x$ means that θ is the angle whose sine is x.

The same relation is expressed by the notation $\theta = \sin^{-1}x$ (read as sine inverse x). So $\sin^{-1}x$ is the angle whose sine is x. In this way, we can define $\cos^{-1}x$, $\tan^{-1}x$, $\operatorname{cosec}^{-1}x$, $\sec^{-1}x$ and

$\cot^{-1}x$. All these are called Inverse Trigonometric Functions or Inverse Circular functions.

Before starting to teach principal values and general value the teacher is advised to take a particular example, say $\sin\theta = \frac{1}{2}$ then $\theta = 30, 150, 390$ etc. so the student is convinced that corresponding to e.g. $\sin\theta = x$, there will be many values for θ satisfying above equation.

Principal Value

If $\sin\theta = x$, we have $\theta = \sin^{-1}x$.

Now, if θ is given, x can have only one value. But if x be given θ (or $\sin^{-1}x$) can have any one value of an indefinite number of angles. All these angles taken together give us the general value of $\sin^{-1}x$. If $\sin\theta = x$, then general value of $\sin^{-1}x$ is $n\pi + (-1)^n$

general value of $\cos^{-1}x$ is $2n\pi \pm \theta$. where $\cos\theta = x$

general value of $\tan^{-1}x$ is $n\pi + \theta$, where $\tan\theta = x$.

Thus all inverse trigonometric functions are many valued functions out of all these angles, the numerically smallest one is called the principal value of the inverse function.

If there are two equal but opposite smallest values (i.e. one +ve and the other -ve), in such a case the positive value is to be considered as the principal value; for example,

$$\cos 30^\circ = \frac{\sqrt{3}}{2} \text{ and } \cos(-30^\circ) = \frac{\sqrt{3}}{2} \therefore \cos^{-1}\frac{\sqrt{3}}{2} = \pm 30$$

In this case principal value of $\cos^{-1}\frac{\sqrt{3}}{2}$ is +30. Evidently the principal value of $\sin^{-1}x$ and $\tan^{-1}x$ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ and that of $\cos^{-1}x$ lies between 0 and π

15.2 Properties of Inverse Trigonometric Functions

Many properties of inverse Trigonometric functions have been discussed in §15.2 of the text-book. Two additional properties which are often required in solving equations involving inverse Trigonometric functions are proved below.

1. $$\sin^{-1}x + \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}].$$

Proof: Let $x = \sin\theta$ and $y = \sin\phi$

$$\Rightarrow \theta = \sin^{-1}x \text{ and } \phi = \sin^{-1}y$$

We know that

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$= \sin\theta\sqrt{1-\sin^2\phi} + \sqrt{1-\sin^2\theta}\sin\phi$$

$$\therefore \theta+\phi = \sin^{-1}[\sin\theta\sqrt{1-\sin^2\phi} + \sqrt{1-\sin^2\theta}\sin\phi]$$

Substituting values, we get

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}].$$

Similarly one can prove

$$\cos^{-1}x + \cos^{-1}y = \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}].$$

Note: The inverse Trigonometric functions and their principal value branches are given in Ex. 15.1 of Text-book.

Equations involving inverse trigonometric functions are solved with the help of Trigonometric relations already discussed in the previous chapter and using properties in §15.2. In solving such equations we have to determine the value of a variable with the help of trigonometric relations.

We shall solve here two equations involving inverse trigonometric functions.

Example: Solve the equation

$$\sin^{-1}\frac{5}{x} + \sin^{-1}\frac{12}{x} = \frac{\pi}{2}$$

Solution: The given equation is

$$\sin^{-1}\frac{5}{x} + \sin^{-1}\frac{12}{x} = \frac{\pi}{2}$$

or $$\sin^{-1}\frac{5}{x} - \frac{\pi}{2} = -\sin^{-1}\left(\frac{12}{x}\right)$$

or $$\sin^{-1}\left(\frac{5}{x}\right) - \sin^{-1}(1) = -\sin^{-1}\left(\frac{12}{x}\right), \quad \because \sin\frac{\pi}{2} = 1$$

or $$\sin^{-1}\left[\frac{5}{x}\sqrt{(1-1)} - 1\sqrt{\left(1-\frac{25}{x^2}\right)}\right] = \sin^{-1}\left(-\frac{12}{x}\right)$$

or $$\sin^{-1}\left[-\sqrt{1-\frac{25}{x^2}}\right] = \sin^{-1}\left(-\frac{12}{x}\right)$$

or $$\sqrt{1-\frac{25}{x^2}} = \frac{12}{x}$$

or $$1-\frac{25}{x^2} = \frac{144}{x^2}$$

or $$\frac{169}{x^2} = 1$$

or $$x^2 = 169$$

or $$x = \pm 13$$

But $x = -13$ makes the L.H.S. of the given equation negative. Hence it is rejected.

$\therefore$ The required solution is $x = 13$.

8. Find, if there is any value of x which strictly satisfies the equation

$$\tan^{-1}\left(\frac{x+1}{x-1}\right) + \tan^{-1}\left(\frac{x-1}{x}\right) = \tan^{-1}(-7)$$

<u>Solution:</u> Applying the formula $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, the given equation becomes

$$\tan^{-1}\frac{\frac{x+1}{x-1}+\frac{x-1}{x}}{1-\frac{x+1}{x-1}\cdot\frac{x-1}{x}} = \tan^{-1}(-7)$$

(The formula is true only when $|x| < 1$, $|y| < 1$

i.e. $\tan^{-1}\dfrac{2x^2-x+1}{1-x} = \tan^{-1}(-7)$

$$\therefore \quad \frac{2x^2-x+1}{1-x} = -7$$

or $\quad 2x^2 - x + 1 = 7x - 7$

or $\quad 2x^2 - 8x + 8 = 0$

or $\quad x^2 - 4x + 4 = 0$

or $\quad (x-2)^2 = 0$

So that $x = 2$.

This value makes both the terms of the left-hand side of the given equation positive; that is, left-hand side is the sum of two positive angles and as such is positive while right-hand side obviously denotes negative angle (between $-\frac{\pi}{2}$ and 0) as its tangent is negative. Therefore, the given equation is not satisfied by $x = 2$. Hence there is no value of x which strictly satisfy the equation.

3. LEARNING OUTCOMES:

a) Essential learning outcomes for all:

The learner should be able to:

i) Define inverse Trigonometric functions

ii) Explain one-one-onto function.

iii) Find principal value of any inverse Trigonometric functions.

iv) Draw the graph of an inverse trigonometric function.

v) Recall all the properties of inverse Trigonometric functions.

vi) Prove the above properties

vii) Solve problems based on the unit.

b) Learning outcomes for the higher group:

i) The Learner with higher ability may be able to solve typical problems based on inverse Trigonometric functions..

ii) He may take less time in solving a problem.

iii) He can make a question using the properties.

4. TEACHING STRATEGIES:

Motivation: A teacher can very well develop lesson making use of previous knowledge of students.

A student is expected to know one-one onto function, Trigonometric ratios and some values of trigonometric ratios.

As explained in content, students can be motivated following steps systematically.

Common Errors: The teacher should explain the difference between $\sin^{-1}x$ and $(\sin x)^{-1}$). Note that $\sin^{-1}x$ is a notation and an angle and $(\sin x)^{-1}$ i.e. $\frac{1}{\sin x}$ is a number. We see that

$$\sin^{-1} x \neq (\sin x)^{-1}$$

even though in notation

$$\sin^2 x = (\sin x)^2, \quad \sin^3 x = (\sin x)^3$$

<u>Graph of Inverse Function</u>

A Teacher should encourage the students to draw graphs of standard functions like $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc. over prescribed intervals. The students should be asked to verify that the graph of $y = \sin x$ and $y = \sin^{-1}x$ are symmetrical about the line $y = x$.

By drawing other graphs, students may see that graphs of all other trigonometric functions and their inverses are symmetrical with each other about the line $y = x$.

Students should also verify that the same conclusion holds for the graphs of $\tan^{-1}x$, $\cos^{-1}x$, $\sec^{-1}x$, $\operatorname{cosec}^{-1}x$, $\cot^{-1}x$.

Hints/Solutions of Problem Exercise 15.1 of the Text-book.

10. If $\cos^{-1}\frac{x}{a} - \cos^{-1}\frac{y}{b} = \theta$ then show that

$$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\theta + \frac{y^2}{b^2} = \sin^2\theta.$$

It is given that $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \theta$

$$\cos^{-1}\left[\frac{x}{a}\cdot\frac{y}{b} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right] = \theta$$

or $$\frac{xy}{ab} - \sqrt{1-\frac{x^2}{a^2}}\cdot\sqrt{1-\frac{y^2}{b^2}} = \cos\theta$$

or $$\frac{xy}{ab} - \cos\theta = \sqrt{1-\frac{x^2}{a^2}}\cdot\sqrt{1-\frac{y^2}{b^2}}$$

Squaring both sides

$$\left(\frac{xy}{ab} - \cos\theta\right)^2 = \left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)$$

$$\frac{x^2y^2}{a^2b^2} - \frac{2xy}{ab}\cos\theta + \cos^2\theta = 1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}$$

$$\therefore \quad \frac{x^2}{a^2} - \frac{2xy}{ab}\cos\theta + \frac{y^2}{b^2} = 1-\cos^2\theta$$

$$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\theta + \frac{y^2}{b^2} = \underline{\sin^2\theta}$$

11. Find the value of

$$\tan\left[\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2}+\frac{1}{2}\cos^{-1}\frac{1-y^2}{1+y^2}\right]$$

$$\text{L.H.S.} = \tan\left[\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2}+\frac{1}{2}\cos^{-1}\frac{1-y^2}{1+y^2}\right]$$

$$= \tan\left[\frac{1}{2}\,2\tan^{-1}x+\frac{1}{2}\,2.\tan^{-1}y\right]$$

$$= \tan[\tan^{-1}x+\tan^{-1}y]$$

$$= \tan\left[\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \frac{x+y}{1-xy} \text{ where } xy < 1$$

ADDITIONAL PROBLEM:

1. Prove that $\cos^{-1}x = 2\cdot\sin^{-1}\sqrt{\frac{1-x}{2}} = 2\cos^{-1}\sqrt{\frac{1+x}{2}}$.

Solution: If $x = \cos\theta$ $\theta = \cos^{-1}x$.

$$\text{Then}\sqrt{\frac{1-x}{2}} = \sqrt{\frac{1-\cos\theta}{2}}$$

$$= \sqrt{\frac{2\sin^2\theta/2}{2}}$$

$$= \sin\frac{\theta}{2}$$

$$\therefore \frac{\theta}{2} = \sin^{-1}\sqrt{\frac{1-x}{2}}$$

$$\theta = 2\sin^{-1}\sqrt{\frac{1-x}{2}}$$

$$\text{i.e. } \cos^{-1}x = 2\sin^{-1}\sqrt{\frac{1-x}{2}}$$

Similarly $\frac{1+x}{2} = \frac{1+\cos\theta}{2}$

$$= \frac{2\cos^2\theta/2}{2}$$

$$= \cos^2\theta/2$$

$\therefore \quad \cos\theta/2 = \sqrt{\frac{1+x}{2}}$

i.e. $\quad \frac{\theta}{2} = \cos^{-1}\sqrt{\frac{1+x}{2}}$

$$\theta = 2\cos^{-1}\sqrt{\frac{1+x}{2}}$$

i.e. $\cos^{-1}x = 2\cos^{-1}\sqrt{\frac{1+x}{2}}$.

2. Show that $\tan^{-1}t + \tan^{-1}\frac{2t}{1-t^2} = \tan^{-1}\frac{3t-t^3}{1-3t^2}$, $t > 0$

Let $t = \tan\theta$ so that $\theta = \tan^{-1}t$

then $\frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta$ or $2\theta = \tan^{-1}\frac{2t}{1-t^2}$.

Also $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

$$3\theta = \tan^{-1}\frac{3t-t^3}{1-3t^2}$$

Now $\theta + 2\theta = 3\theta$

i.e. $\tan^{-1}t + 2\tan^{-1}\frac{2t}{1-t^2} = \tan^{-1}\frac{3t-t^3}{1-3t^2}$

3. Write the following function in the simplest form

$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$

1st Method:

$$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right]$$

$$= \cot^{-1}\left[\frac{(\sqrt{1+\sin x})^2+(\sqrt{1-\sin x})^2+2\sqrt{1+\sin x}\,.\sqrt{1-\sin x}}{(\sqrt{1+\sin x})^2-(\sqrt{1-\sin x})^2}\right]$$

$$= \cot^{-1}\left[\frac{1+\sin x+1-\sin x+2\sqrt{1+\sin x}.\sqrt{1-\sin x}}{1+\sin x-1+\sin x}\right]$$

$$= \cot^{-1}\left[\frac{2+2\sqrt{(1+\sin x)(1-\sin x)}}{2\sin x}\right]$$

$$= \cot^{-1}\left[\frac{2(1+\sqrt{1-\sin^2 x})}{2\sin x}\right]$$

$$= \cot^{-1}\left[\frac{1+\cos x}{\sin x}\right]$$

$$= \cot^{-1}\frac{2\cos^2 x/2}{2\sin\frac{x}{2}\,.\,\cos\frac{x}{2}}$$

$$= \cot^{-1}\frac{\cos x/2}{\sin x/2}$$

$$= \cot^{-1}\cot\frac{x}{2}$$

$$= \frac{x}{2}$$

ALTERNATIVE METHOD:

$$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$$

$$= \cot^{-1}\left[\frac{\cos\frac{x}{2} + \sin\frac{x}{2} + \cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2} - \cos\frac{x}{2} + \sin\frac{x}{2}}\right]$$

$$(\because\ 1+\sin x = (\cos\tfrac{x}{2} + \sin\tfrac{x}{2})^2$$

$$1-\sin x = (\cos\tfrac{x}{2} - \sin\tfrac{x}{2})^2$$

$$= \cot^{-1}\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} = \cot^{-1}\cot\frac{x}{2} = \frac{x}{2}$$

4. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, show that $x^2+y^2+z^2+2xyz = 1$.

<u>Solution:</u> Let $\cos^{-1}x = A$, $\cos^{-1}y = B$, $\cos^{-1}z = C$

Then $x = \cos A$, $y = \cos B$ and $z = \cos C$.

Then $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

i.e. $\cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$

i.e. $\cos^{-1}[xy-\sqrt{1-x^2}\sqrt{1-y^2}] = \pi - \cos^{-1}z$

$\because \cos^{-1}x + \cos^{-1}y = \cos^{-1}[xy-\sqrt{1-x^2}.\sqrt{1-y^2}]$

i.e. $\cos^{-1}[xy-\sqrt{1-x^2}\sqrt{1-y^2}] = \pi - \cos^{-1}z$

$\cos^{-1}[xy-\sqrt{1-x^2}.\sqrt{1-y^2}] = (\pi-\cos^{-1}z)$

$= \cos^{-1}(-z)$.

$\therefore xy-\sqrt{1-x^2}.\sqrt{1-x^2} = \cos\cos^{-1}(-z)$

$(xy+z)^2 = [\sqrt{1-x^2}.\sqrt{1-y^2}]^2$

$\therefore x^2+y^2+z^2 + 2xy = 1$

ALTERNATIVE METHOD:

The identity $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C =$ holds when $A+B+C = \pi$.

The teacher may ask the student to prove this identity and use it in solving the given problem.

5. ALTERNATIVE METHOD FOR THE WORKED OUT EXAMPLE IN 15.7, PAGE 297 OF THE BOOK:

Example: Show that

$$\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right] = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$$

Solution:

$$\text{L.H.S.} = \tan^{-1}\left(\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right] \times \left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]\right)$$

$$= \tan^{-1}\left[\frac{1+x^2+1-x^2+2\sqrt{1-x^4}}{1+x^2-1+x^2}\right]$$

$$= \tan^{-1}\left[\frac{2+2\sqrt{1-x^4}}{2x^2}\right] = \tan^{-1}\frac{1+\sqrt{1-x^4}}{x^2}$$

Let $x^2 = \sin\theta$. Then

$$\text{L.H.S.} = \tan^{-1}\frac{1+\cos\theta}{\sin\theta}$$

$$= \tan^{-1}\frac{2\cos^2\theta/2}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \tan^{-1}\frac{\cos\theta/2}{\sin\theta/2}$$

$= \tan^{-1} \cot \frac{\theta}{2}$

$= \tan^{-1}[\tan(\frac{\pi}{2} - \frac{\theta}{2})]$

$= \frac{\pi}{2} - \frac{\theta}{2}$

$= \frac{\pi}{2} - \frac{1}{2} \sin^{-1} x^2$

$= \frac{\pi}{2} - \frac{1}{2}[\frac{\pi}{2} - \cos^{-1}x^2]$

$= \frac{\pi}{2} - \frac{\pi}{4} + \frac{1}{2} \cos^{-1}x^2$

$= \frac{\pi}{4} + \frac{1}{2} \cos^{-1}x^2$

= R.H.S

.5.. CHAPTER TEST

(a) Oral Test

1. What is the meaning of Principal value?

2. If $\sin\theta = x$, then what is the value of $\sin^{-1}x$?

3. What is the difference between $\sin^{-1}x$ and $(\sin x)^{-1}$?

4. If $\sin\theta = x$, what will be the value of $\operatorname{cosec}^{-1} \frac{1}{u}$?

5. Give the values of

   a) $\sin(\sin^{-1}\theta)$

   b) $\cos(\cos^{-1}\theta)$

   c) $\tan(\tan^{-1}\theta)$

6. What are the Principal value of

   a) $\sin^{-1} \frac{1}{2}$

   b) $\cos^{-1} \frac{1}{\sqrt{2}}$

   c) $\tan^{-1} 1.$

(b) <u>Written Test</u>

1. Prove that $\cot^{-1}x + \tan^{-1}\frac{1}{x} = \frac{\pi}{4}$.

2. Use $\tan(\theta+\phi) = \dfrac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}$ to obtain the formula for $\tan^{-1}x + \tan^{-1}y$.

3. Using $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ prove that $3\cos^{-1}x = \cos^{-1}(4x^3-3x)$.

4. Prove that
$$\tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{2}.$$

5. Show that
$$\tan^{-1}\frac{m}{n} - \tan^{-1}\left(\frac{m-n}{m+n}\right) = \frac{\pi}{4}$$

6. Show that
$$\cos^{-1}\sqrt{\frac{a-x}{a-b}} = \sin^{-1}\sqrt{\frac{x-b}{a-b}} = \cot^{-1}\sqrt{\frac{a-x}{x-b}}$$

7. If $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$, then prove that
$$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha.$$

8. Solve the equations:
$$\cot^{-1}x + \tan^{-1}3 = \frac{\pi}{2} \qquad (\text{Ans: } x = 3)$$

9. $\tan^{-1}\dfrac{x-1}{x-2} + \tan^{-1}\dfrac{x+1}{x+2} = \dfrac{\pi}{4}$ (Ans $x = \pm\dfrac{1}{\sqrt{2}}$

10. If $\phi = \tan^{-1}\dfrac{x\sqrt{3}}{2k-x}$ and $\theta = \tan^{-1}\left(\dfrac{2x-k}{k\sqrt{3}}\right)$, prove that one of $\phi-\theta$ is 30°.

ADDITIONAL EXERCISE:

1. Find the Principal value of each of the following:

(a) $\sin^{-1}\left(-\frac{1}{2}\right)$ Ans (a) $-\frac{\pi}{6}$

(b) $\sec^{-1}(-\sqrt{2})$ (b) $\frac{3\pi}{4}$

2. Find the value of

(a) $\cos\left(\sin^{-1}\frac{1}{4} + \cos^{-1}\frac{1}{4}\right)$ Ans: (a) 0

(b) $\sec^{2}(\tan^{-1}2) + \operatorname{cosec}^{2}(\cot^{-1}3)$ (b) 15

3. Prove that

(a) $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

(b) $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\left(\frac{31}{17}\right)$

(c) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$

$= 2(\tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$

(d) $\tan^{-1}\left(\frac{3}{4}\right) = 2\tan^{-1}\frac{1}{3}$

4. Solve the equations:

(a) $\sin^{-1}\frac{5}{x} + \sin^{-1}\frac{12}{x} = \frac{\pi}{2}$ Ans: (a) $x = 13$

(b) $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$ (b) $x = \frac{1}{6}$

5. Solve the equation:

$\sec^{-1}\frac{x}{a} - \sec^{-1}\frac{x}{b} = \sec^{-1}b - \sec^{-1}a.$ Ans: $x = ab$

6. If $\sec\theta - \operatorname{cosec}\theta = \frac{4}{3}$,

show that $\theta = \frac{1}{2}\sin^{-1}\frac{3}{4}$

LIST OF REFERENCE BOOKS

1.	Plane Trigonometry	By S.L. Loney
2.	Trigonometry	S.Narayanan
3.	Mathematical Handbook Elementary Mathematics	By M.VYGODSKY Mir Publishers, Moscow
4.	A Text Book on Plane Trigonometry	By S.P. Nigam and B.S. Tyag
5.	A New Book of Mathematics	By C.S. Sarana, R.G. Gupta, F.K. Garg Arya Book Depot, Karol Bagh, Delhi
6.	Trigonometry	By Prof. P.N. Chattorji Raj Hans Prakashan Mandir, Meerut (U.P.).

CHAPTER 16

FREQUENCY TABLES

1. INTRODUCTION

The origin of the word "Statistics" may be traced to the compilation by governments, since very early times, of a record of population and wealth to help in formulation of policies. Since the compilation work was carried out by the government (or, the state) the term "Statistics" was used to describe the results of that activity, the word "Statistics" being derived from the Latin word for the State.

In India too, statistical records are known to have been in existence more than 2000 years ago during the reign of Chandragupta Maurya (324-300 B.C). Kautilya's Arthashashtra talks of statistics of births and deaths. There is a mention of maintenance of statistical records for administrative purposes in Ain-i-Akbari (1556-1605 A.D.)

Over a long period of time the area of maintenance of numerical records has gradually extended to cover very different types of data. Thus we have trade statistics, labour statistics, health statistics, educational statistics, etc. The methodology of statistical analysis too has undergone a big change with the development of its theoretical base.

The theoretical foundations of the modern science of statistic were laid by R.A.Fisher. In India, the development of statistical knowledge and its utilisation in planning owes a great deal to the two pioneers : P.C. Mahalanobis and P.V. Sukhatme.

The use of statistics is no longer confined to the collection and compilation of data. Today statistics finds its application in such diverse fields as the control over the quality of industrial production, the diagnosis of diseases in medicine, the selection of proper seeds and fertilisers to improve agricultural production, the development of marketing strategies in commercial activities, and even in the prediction of election results!

This chapter in the first exposure of the student to statistics as a very useful and significant component of applied mathematics. As such, it has no direct relation to the preceding succeeding chapters of the book. The coverage is confined to a study of frequency tables as an important tool in presenting raw data in a form which makes it easier to grasp its salient points, and to the use of measures of location and dispersion as summaries the information contained in the data on quantitative variables.

A number of technical terms have been introduced during the course of the discussion. These are : family budget survey, cost of living index, age-distribution, sex-ratio, age-specific death rates. Brief description, or definition, has been given for each of them. Mention has also been made of a number of agencies (Census Organisation, National Sample Survey, Ministry of Labour) which are engaged in the collection of statistical information. These can be used not only to increase the general knowledge of the student but also to make him aware that the use of statistics occupies an important place in the management of the country's affairs.

The contents of this chapter have been developed with a view to clarifying why we do certain things instead of merely describing what we do. The computations that are to be made are not to be regarded as mere arithmetic exercises but as being done with the purpose of drawing some conclusions. The essence of learning statistics at this level is to develop the ability to make an intelligent use of numerical data. These considerations may be kept in mind while teaching the material of this chapter.

2. CONTENT ANALYSIS

In this section the number of each sub-section is in accordance with the text book.

16.1 : Raw Data

This section introduces the student to the fact that statistics deals with information given in numerical form. It describes that such information is collected by different methods (census, survey, routine recording) and has to be processed to make it intelligible.

Mention has been made of some agencies engaged in the collection of data. Besides a number of technical terms (family budget survey, cost of living index, age-distribution, sex ratio, age-specific death rates) have been introduced. This increases the student's general knowledge and makes him aware of the diversity of areas where statistics is used.

An additional idea, usually not mentioned, is that of scrutinising all numerical data, so that, a student becomes as adept in dealing with the language of numbers as he is with ordinary language. The need for scrutinising all numerical data is established by means of two examples where the information as printed had errors.

16.2 : Variables of Observation

The term "variable of observation" is new and has been used to draw attention to the fact that what we deal with is not just some abstract variable but is a result of some actual process of observation.

The distinction between a variable and its values is made, as also between "possible values" and "observed values".

16.3: Qualitative and Quantitative Variables

The distinction between qualitative and quantitative variables is based on the nature of their values. Examples illustrate the difference.

16.4: Units of Observation

This concept is new, in that it is not usually stated explicitly but rather left to one's own way of understanding it. It is not to be confused with unit of measurement.

Sections 16.2, 16.3 and 16.4 constitute one continuous chain of ideas which form the basis of understanding the treatment of frequency distributions.

16.5: Frequency Tables

This section deals with the following main ideas:

i) The utility of the frequency table as a means of suitably presenting raw data.

ii) The structure of a frequency table in terms of given units of observation and a given variable of observation.

iii) The equivalence of the terms frequency table and frequency distribution.

A novel idea is a discussion of the manner in which the classes of a frequency table are described by the values of a continuous variable. The two examples considered show that an analysis of the method of recording the observations, or of the

way classes are described may make it clear that the classes are properly defined even if they do not appear to be so.

The problem of how to determine the width of an interval, in case of frequency tables involving a continuous variable, has not been discussed. It may be taken up by the teacher if time is available, or interest is shown by the student.

16.6: Construction of Frequency Tables from Raw Data

This section describes the use of the well known method of tally marks in construction of frequency tables from raw data.

16.7: Relative Frequency Table

This section defines relative frequency and the relative frequency table resulting from it.

The usefulness of a relative frequency table in.

i) understanding and assimilating the information contained in a frequency table.

ii) comparing two frequency distributions, is brought out through a discussion of some examples. What is thereby emphasised is that the relative frequency distribution is an important tool in understanding and analysis of numerical information.

16.8: Graphical Presentation of Frequency Distributions

The Section deals with the methods of construction of bar diagrams and pie diagrams.

The importance of these methods has been brought out through examples. Attention has been drawn to the fact that the same bar diagram can be used to represent a frequency distribution with actual frequencies or with relative frequencies. In case of comparison of two frequency distributions the usefulness of graphical presentation through bar diagrams using relative frequencies has been brought out.

16.9: Measures of Location and Dispersion

This section only draws attention to the fact that these measures are relevant only in case of frequency distributions based on values of a quantitative variable, and that they are really meant to provide some sort of a summary of the information contained in a frequency table.

16.10: Measures of Location

(a) The Arithmetic Mean

This section gives the basic definition of the arithmetic mean and describes how, from this definition, the mean can be considered to be a central value.

The section also describes the methods of calculating, or approximating, the value of the mean from frequency tables.

The section has some exercises which illustrate the behaviour of the mean under linear transformations.

(b) The Median

This section defines the median in a new manner. When this definition is applied to different situations, where the total number of observations is odd or even, we get the usual rules for obtaining the median in such cases.

The new definition of the median makes the determination of the median unambiguous.

The section also describes the methods of obtaining the median, exact or approximate as the case may be, from frequency tables.

The section also describes in what sense is the median to be regarded as acentral value.

16.11: The introduction to this section tries to describe the idea of dispersion (or scatter) of a number of values of a quantitative variable of observation about a fixed value. Sub-section (a) is devoted to variance and standard deviation; and (b) to the study of mean deviation.

(a) Variance and standard deviation

The presentation follows a method different from the usual method. It builds up the definition of the variance step by step by starting with the individual deviations $x_i - x$ and then combining them into a single number measuring the dispersion.

The section also emphasises that the calculation of the variance should be done by using the identity $\sum_i (x_i - \bar{x})^{-2} = \sum_i x_i^2 - \left(\frac{\sum x_i}{n} \right)^2$. The use of the right hand side of this identity is to be preferred as it reduces the amount of computation and the chances of making errors.

The use of linear transformation of the values to reduce the amount of calculations is also described.

(b) <u>Mean deviation about the median</u>

The definition of the mean deviation about the median is given and there is a discussion why it could also be considered to be a measure of dispersion.

A simpler method of calculating the mean deviation about the median is given which is applicable where the individual observations are known.

The idea of using a measure of location along with a suitable measure of dispersion is mentioned.

3. LEARNING OUTCOMES

(a) Essential learning outcomes for all

(i) To become "numerate", that is to say, to become able to read, scrutinise, and understand information presented in numerical form, and to realise the need for processing of raw data.

ii) To learn the basic structure of a frequency table in terms of variables of observation and units of observations, and to construct frequency tables from raw data.

iii) To extract useful information from frequency tables and to compare two frequency distributions with the help of relative frequencies.

iv) To acquire skills in the calculation of the mean, median, variance, standard deviation, and mean deviation, and understand the utility of these measures of location and dispersion as summaries of the information contained in frequency tables, or raw data, based on quantitative variables.

(b) Learning outcomes for the higher groups

i) To appreciate the usefulness and relevance of statistical data and analysis in widely differing fields.

ii) To develop the faculty of quantitative reasoning.

4. TEACHING STRATEGIES

16.1 RAW DATA

This section is intended to introduce the student to the concept of statistics as making use of information provided in numerical form. Some examples are briefly given in the first paragra and may be supplemented by other examples by the teacher.

The basic idea to be grasped here is that all numerical information that comes to us has first been collected. What is collected is our raw data but what is presented is something different. Unless the raw data is suitably presented it cannot be made use of, specially if the size of the raw data is large. The example of the Census data (Table 16.1) was specially selected to drive home the fact that raw data may be unmanageable and un-intelligible unless it is suitably processed and presented in a different form.

Once we have the raw data in a suitably presented form we have to learn how to make use of it. For example, Table 16.1, at first glance is only a set of numbers classified under different headings. But what is the information being conveyed by it ? We get an idea of information can be extracted from the table, from the calculations described on page 303. The process is similar to what happens when we read a text written in ordinary language. The hidden meaning is grasped only after a lot of thinking over the text and after comparing and cross-checking it with previously known facts. In the same way here too information is to be extracte by the mind from the numerical data, and hence, the mind must be trained to read and understand the language of numbers. Just as we use the term "literate" to denote a person who is able to read and understand a text written in ordinary language, we may

use the term "numerate" to denote a person who is able to read and understand the language of numbers.

The examples and suggested exercises given on page 303 showing how further information can be extracted from Table 16.1, can be supplemented by further exercises. For example, we may compare the percentage of rural to total population in different age-groups, we may compare the age-distributions of males and females in the rural areas or, the age-distribution of males in rural and urban areas, etc. The students could even be asked to extract whatever additional information they can think of.

The next point which has been stressed is that all data, in the form it has been presented, must be scrutinised for discrepancies and efrors, and efforts made to remove them whenever possible. In the census table three of the figures were obviously wrong. The student should be encouraged to understand the method of detection which was possibly used. The best thing would be to replace the corrected figures by the original figures (reporduced at the foot of the table) and then ask the students to see why these original figures were found to be incorrect ? If the student is able to find out, he would also be able to carry out the corrections. The thing to note is that the entries in Table 16.1 are being totalled in different ways. Inside each age-group, and for "all-ages" also, the first row and column are formed by the totals of the entries in the other rows and columns. Again, the entries in the "all-ages" section are the totals of the corresponding entries in the sub-sections. For example, for the column "Male" we have

47,016,421 = 13,724,165 + 6,061,626 + ...+ 3,723

The entry 270,828** in the Female Column on page 302 was printed as 270,823 in the original census publication. If we take that value then the total of rural and urban values in that column is 1,916,479 + 270,823 = 2,187,302 and not 2,187,307, indicating the presence of an error. By cross checking against all other sub-totals one comes to the conclusion that the value should be 270,828. The other two errors were also corrected in the same manner. It is interesting to note that some more errors have crept in, which are not in the original census table. These are to be corrected as follows:

1) The total population of all ages is printed as 88,311,144 but should be 88,341,144.

2) In the 30-34 age group the total number of females should be 2,897,439 and the number of urban females should be 354,534.

3) In the 45-49 age group the total number of males should be 1,984,626.

4) In the 50-54 age group the total rural population should be 3,1414,675.

5) In the 70+ age group the total urban population should ~~xx~~ be 237,435.

Similarly, in Table 16.2, the error which has been pointed out in the text should be brought to the notice of the students and they should be encouraged to guess why that figure has been declared to be wrong. The answer is from the theory of average

and ratios. If the percentage of boys in one class-room is P_1 and P_2, then the percentage p of boys taking the students of both the class rooms together must have a value between p_1 and p_2. Algebraically the result is:

$$\text{If } \frac{a}{b} < \frac{c}{d} \text{ then } \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

This can be proved in many ways. One proof is given below:

Put $\frac{a}{b} = 1$, $\frac{c}{d} = m$ so that $1 < m$,

$$\text{then} \quad \frac{a+c}{b+d} = \frac{bl + dm}{b + d} < \frac{bm + dm}{b + d} = m = \frac{c}{d},$$

$$\text{and} \quad \frac{a+c}{b+d} = \frac{bl + dm}{b + d} > \frac{bl + dl}{b + d} = 1 = \frac{a}{b}$$

If the students are not familiar with this they can be asked to prove it as an exercise. Now, Table 16.2 gives for each age-group the ratio of male deaths to male population, ratio of female deaths to female population and the ratio of total deaths to total population. Hence, the values entered in the "Both" column should lie between the values in "Males" and "Females" column. Except for the value indicated, all other values satisfy this condition. That is why the error was spotted. However, in this case, the error cannot be corrected unless one knows the number of males and females in the aged-group 40-44. Note that even if one did not know anything about how death rates are calculated, and particularly that they are ratios, one could still see that except for the 40-44 age group the values in the "Both" column lie between those in the "Males" and "Females" column. This would set one thingking to see if this was really unused and could this be an error. In fact, the students could first be

asked to see why the value marked with an asterisk suggests that there is an error. Once, they spot that it does not follow the regular pattern then they can be asked to find out if they can definitely say that it is an error and finally asked if it is possible to correct it and if not, what additional information is needed to make the correction. In many cases, a scrutiny of any numerical data can reveal some things falling outside a general pattern and those have always to be looked into to see if they were due to errors of recording, printing or calculating. The ability to spot any such "abnormalities" in the data is an essential part of the training in using statistical data and analysis.

The students should be encouraged to think of the reasons that could explain why the death-rates in the different age-groups are different for males and females, and why the death-rate is so high (for both males and females) in the 1-4 age-group. The possible reasons are:

(i) A large number of deaths take place in the 1-4 age-group because of lack of proper maternity and early child-care services in rural areas.

ii) For the same reason death-rate for women in the age-groups 15-19, 20-24, 25-29, 30-34 is high, as women in these age-groups run the risk of death during child-birth.

iii) Death-rate in age-groups 1-4, 5-9 is higher for women possibly because female children receive less care and attention as compared to male children.

iv) Death-rate for males is more, as compared to females, in the higher age-groups because the males are subject to greater stress.

16.2 Variables of Observation

This section introduces the idea of a variable of observation. Notice that we are talking of a variable of observation and not just a variable. The idea is that in our raw data, whatever it may be, every single entry reports an observation that has been made. The raw data is a result of some observation dprocess; the measurement of an area, finding out the religion of an individual the noting of a birth or a death, etc. Before the raw data is collected, the variable, or variables, about which observations are to be obtained have to be decided upon, as also the manner in which the process of obtaining the observation and recording the result is carried out. In each case we are thus concerned with a specific variable that was observed and not with the general notion of a variable as in mathematics.

The distinction between the variable of observation and its values is very important and should be clearly established in the mind of the student. For, the two are normally confused and one does come across statements where male and female, or Hindu, Christian, etc., are stated to be different variables.

The distinction is more or less like that between a function and its value in mathematics. There, the distinction between sin x as one of the trignometric functions and sin x as the value of that function for an angle of magnitude x is clear. So much so that we often use the same notation sin x for both.

It would be better to ask the student to try to locate what were the variable or variables of observation in any example that he comes across either in the text or in the discussion in the class room. He should be encouraged to make this a habit. This is essential because the numbers or numerical results that one deals with in statistics are not just abstract numbers, nor are the operations we perform in statistics merely arithmetical operations. We are concerned with actual situations. When we have data on heights we are not just looking at some numbers as length measurements. But every height recorded in the raw data is the height of some individual identified as a member of some distinct group.

Each variable is really completely known by the set of its possible values - some, or all of these values will enter in the recorded raw data. Thus, the possible values of the age variable may be numbers 0, 1, 2,..... . if we are recording the age in completed years, or in completed months or weeks. When we observe a particular group of individuals all these numbers will not appear in the raw data but only some of them. For example, if one records the ages of the students in the class, only those numbers will appear in the raw data which correspond to the ages of the students in the class. Hence, the distinction between possible values of a variable and its recorded values.

The idea of a variable of observation is connected with that of a unit of observation discussed in Section 16.4. The two ideas go together and can only be fully understood in terms of each other.

16.3 Qualitative and Quantitative Variables

The variables are divided into different types according to the nature of their values. We have only considered two broad types here, qualitative and quantitative. The important consideration here is that what is really relevant is the nature of the values and not the way they are recorded and this difference has to be clearly explained to the student. With modern methods of analysis and specially with the increasing use of computers many qualitative variables are coded as numbers but that does not make such variables quantitative variables.

The students should be encouraged to think of variables themselves and to classify them as qualitative or quantitative. Since the distinction can be made only by the description of the values of the variable one sees the importance of always thinking of a variable in terms of its values as emphasised in Section 16.2.

The distinction between these two types of variables is also important because, as appears later in the text, some methods of of analysis can be used with only one of the two types of variables.

16.4 Units of Observation

As we have mentioned earlier, the raw data talks of some quality (sex, religion, political parts, place of residence, etc.) or some magnitude (height, weight, salary, area, etc.) i.e. represents the values of the variable of observation, and each of these values results from some observation being made - a plot of land is measured, an individual's relgion is noted, etc. Thus every recorded value of a variable is attached to a source from where this value comes. It is this source which has been called the unit of observation. Thus, if the raw-data consists of a number of recorded values of a variable there will be as many units of observation from where these values have been obtained. While looking at any data these two aspects, the variable of observation and the unit of observation, have to be clearly understood. Only then it will be possible to properly analyse the data and interpret the information obtained from it.

The unit of observation is not to be confused with the unit of measurement. The latter is used to describe the scale of measurement of a quantitative variable. It only tells us whether the height will be recorded in cms or mms, the weight in grams of kilogms., etc.

16.5 Frequency Tables

One should begin by making the students assimilate the form in which a frequency table occurs by using the two examples in the text In both we have in one column (the first) a grouping of the values of a variable into different classes and in another column the frequency of each class of this grouping. The students must be asked to "read" these two frequency tables. In the first table - Table 16.3 - suing only columns 2 and 3 we "read" that 7116591 farmers owned land which was less than 0.5 hectares in area, 2986638 farmers had land with area between 0.5 hactares and 1 hectare, ... and so on fill the last row saying that 1406 farmers owned land which was more than 50 hectares. In this example, the first column (i.e. Col. 2) refers to the value of a quantitative variable of observation - the area - with the hectare being the unit of measurement, and the second column (i.e. Col.3) refers to the units of observation - the farmers - from whom the values of the variable of observation were obtained. The same is the case with Table 16.4 where the first column refers to the variable of observation - the age - with the year being the unit of measurement, and the second column refers to the units of observation - the inhabitants of Uttar Pradesh.

Table 16.3 - where the title is given as it was in the Census Tables - could have been called (restricting ourselves to column 2 & 3 only) the frequency table (or frequency distribution) of land-holding by area. (or. of farmers by area of land held). Similarly, Table 16.4 is called frequency table of inhabitants of Uttar Pradesh by age. We could also have called it a frequency distribution instead of a frequency table.

The important thing to note and understand here is that the title itself, if properly written, describes how the table was constructed. In every frequency table it is the total frequency which is distributed into various classes. In other words the units of observation are being assigned to different classes. The classes are defined in terms of the values of the variable.

In every frequency table that the student comes across he should first clearly identify the unit of observation and the variable of observation and then only proceed further.

The utility of a frequency table should be brought out by giving exercises to the student to extract information from it. The exercises will highlight the fact that the information thus extracted from the table could not be obtained by looking at the raw data and if calculated from the raw data it would have taken a lot of time and effort. With a frequency table some of these conclusions can be drawn by carefully looking at the table, some by doing some quick calculations. The table presents the raw data in a form in which further calculations can be quickly done and a number of questions answered. That precisely is its utility.

As a simple example, I can easily obtain the percentage of the population below 30 years of age from the Census table. It would not have been possible to get it easily from the raw data. However, I can not easily get the percentage of the population with age less than 33 years due to the manner in which the values of the age variable have been grouped into classes in the frequency table. This problem should be posed before the

students and they should be asked to suggest a solution. The better students would perhaps be able to suggest the solution as one of approximating the percentage from grouped data after they have studied the methods of calculating the mean and median from grouped data. The teacher should recall this problem to the students after they have completed the study of calculation of mean and median from grouped data and see if they can now suggest a solution. The teacher should provide the solution only after this stage and not earlier.

The method of constructing a frequency table has two stages At first the possible values of the variable are grouped into classes. The first idea to bring out is that a class may be defined by a single value or a number of values. The two examples in the text have more than one value in a class. The teacher should present an example of a frequency distribution where each class is defined by a single value. The students should also be given the following example from the results of the last general election presented in the newspapers.

FINAL LOK SABHA ELECTION SCORE CARD

Parties		No. of Seats
Congress		191
Janta Dal		141
DMK		-
Telegu Desam		2
BJP		86
CPM		32
CPI		12
I & O*		59
	Total	523

*Independents and Others

(Source: Times of India, New Delhi, December 1, 1989)

This is also a frequency table where the unit of observation is the individual declared by the Election Commission to have been elected to the Lok Sabha, and the variable of observation is the party on whose ticket he has been elected. The students may be asked to note that all the classes except the last are defined by a single value of the variable of observation, but the last one consists of more than one value. Thus, in the same table we can have some classes defined by a single value and some others by more than one value of the variable of observation. Such a situation usually occurs in frequency tables depending on qualitative variables where the more important classes are defined through single values and the remaining values are clubbed together in the class called "others". For example, when we have a frequency table of individuals by religion ~~x~~ we may find the classes defined as Hindu, Muslim, Sikh, Christian, Others. Our choice of the manner of defining our classes depends on what important idea we want to convey. For example, the above table of election results could also have been presented as

Parties		No. of Seats
Congress		191
National Front		143
BJP		86
CPM		32
CPI		12
I & O		59
	TOTAL	523

Here we have created another class called "National Front" defined by more than one value of the variable, by considering Janta Dal, DMK and Telegu Desam MP's as belonging to this class.

The second point to be highlighted is that the classes into which the values of the variable of observation have been grouped constitute what may be called as "mutually exclusive and exhaustive system of classes" This idea has been described in the third paragraph on page 308 The idea is easy to grasp but problems arise when one is dealing with quantitative variables which are continuous. How does one describe the classes in such cases. Tables 16.3 and 16.4 illustrate the difficulty. In the first some values occur in more than one class and in the second some values do not occur at all. In both cases we have explained to the student how to interpret the meaning of the classes so that they are seen to have been properly constituted. In both the tables the class a-b is understood to mean values equal to or more than a and less than b. This interpretation is acceptable because of the way the table on land area has been presented with the beginning class described as "below 0.5" and the last one as "50.0 and above", and because of our knowledge about the manner in which age is usually recorded. But there are other types of cases also. For example, the life insurance companies want the age to be recorded as "age nearest birth day".

In this case too the recorded raw data will show the age in whole numbers But now the class (or age-group) 10-14 will not represent "age 10 or more but less than 15" as in the census example. Under the new system of recording the observation on the age-variable, any person whose age is 9½ years or more but less than

$10\frac{1}{2}$ years will be shown as having age equal to 10 years. So that the age-group 10-14 now represents ages of $9\frac{1}{2}$ years or more but less than $14\frac{1}{2}$ years. Our frequency table may still show the classes as in the census example (Table 16.4) but the classes will not stand for the same set of values as earlier. The age-group 10-14 will thus stand for "age 10 years or more but less than 15" if age is recorded in completed years, but will stand for "age $9\frac{1}{2}$ years or more but less than $14\frac{1}{2}$ years" if age is recorded as age nearest birthday.

Notice that the classes in Table 16.4 could also have been shown as 0-10, 10-15, 15-20,--------------65-70, 70- , with the clear understanding that in every group the right hand limit was to be excluded. So that the group 15-20 stands for "age 15 years or more but less than 20 years". But if age is recorded as "age nearest birthday" then the classes can be shown as 0-9, 10-14, 15-19,....... ... but not as 0-10, 10-15, 15-20-------- In the second representation the class 10-15 would stand for $9\frac{1}{2}$ years or more but less than $15\frac{1}{2}$ years, and the class 15-20 would stand for $14\frac{1}{2}$ years or more and less than $15\frac{1}{2}$ years would be common to the classes 10-15 and 15-20. If in this case we want to represent the classes without gaps we will write 0-9, 10-14, 15-1' .... as 0-9,5, 9.5-14.5, 14.5-19.5, ... with the convention that the right hand side is to be excluded so that 9.5-14.5 stands for age 9.5 years or more but less than 14.5 years.

As you may have noticed the problem arises only when we are dealing with continuous variables. Such variables can conceivably take any value between two given values. As against it, discrete variables have values which can be labelled as 1st, 2nd, 3rd, ..., so that for any value one can indicate what the next value will be,

something which is impossible to do in the case of continuous variables. Yet, at the time of observation every continuous variable becomes a discrete variable due to the limit of accuracy of measurement; length is a continuous variable but if we measure it upto centimeters, rounding off the fractional part, our observations yield discrete values. If we measure it upto millimetres, i.e., upto one decimal point in centimetres, again the observations yield discrete values. No matter upto what high level of accuracy we make the measurement some fractional part will still have to be rounded off. It is in this sense that we say that every continuous variable becomes a discrete variable at the time of making observations on it.

It is because of this that certain authors make a distinction between class-marks and class-limit in the case of continuous quantitative variables. But we need not go into it at this stage. It is enough if the student is able to verify that the classes in a frequency table have been properly defined.

16.5 Construction of Frequency Tables from Raw Data

The last stage in the construction of frequency tables is the use of tally marks. The students must understand that this is a procedure for making the frequency table conveniently and with less chances of error. I have noticed that many a time the students show the tally marks in their exercise books but have not used them properly. They first count from the raw data the number of units of observation falling in a particular class and then put the requisite number of tally marks in their answer sheet. The answer looks all right but the procedure is totally

wrong. The very purpose of using tally marks is lost. It must be emphasised that we put the tally marks one by one as we read through the values of the variable of observation. It would be a good idea to let the student make a frequency table by counting from the raw data and by the use of tally marks in the proper manner. If the raw data has, say, 100 observations the student would see easily why the method of tally marks is to be preferred.

There is one problem regarding making of frequency tables from raw data which we have not touched upon. That is, in case of quantitative variables, how does one determine the width of the intervals which from the classes into which the values of the variable are divided. Should the intervals be of equal or unequal size ? Should they be of small width or large width ? It is not possible to go into the details of this question at this stage. It would be sufficient at this stage to highlight the following aspects. As far as possible, one tries to have intervals of equal width but there is nothing wrong in having unequal width intervals. In fact in some cases one deliberately chooses intervals of unequal width. As an example of a frequency table where unequal class intervals have been chosen the student's attention should be drawn to Table 16.3. Here the first two class intervals are 0.5 hectares in width, the next four are of 1 hectare width followed by one interval of 5 hectare width and four of 10 hectare width, and the last interval is what we call "open ended" that is an interval whose width is indeterminate. If we had kept all intervals of 10 hectare width, keeping the last one open ended our frequency distribution would be as follows:

Size-class (in hectares)		No. of holdings	%age
Below 10.0		15046791	99.26
10.0 - 20.0		94727	9.62
20.0 - 30.0		11752	0.08
30.0 - 40.0		3198	0.02
40.0 50.0		1058	0.007
50.0 and above		1406	0.009
	TOTAL	15158932	99.996

From such a table we could only conclude that 90% holdings are below 10 hectares and almost all (i.e. 99.26 + 0.62 = 99.88%) are below 20 hectares. And thereby the very important information provided by Table 16.3 that almost half (46.9%) the holdings are less than half a hectare and nearly two third (46.9 + 19.7=66.6%) are less than one hectare would have been lost. The particular choice of unequal class intervals in Table 16.3 was made because we know that a very large number of peasants in the country have very small pieces of land and we wanted this aspect to be brought out very accurately to help in evolving proper policies. Again, if we look at columns 3(a) and 4(a) of Table 16.3 we find that persons holding less than half a hectare each constitute 46.9% of the population but the land owned by them is only 8.6% of the total land. At the other end of the table persons having more than 50 hectares of land form only 0.009% of the population (i.e. 1 in 10,000) but own 0 8% of the land. In between also we find, for example that persons with land between 3 and 4 hectares own 10% of the land but constitute only 3.5% of the population. Thus, Table 16.3 clearly reveals the glaring inequality in land ownership

in the countryside. If we do the same claculations in the modified table we find that 99.26% of the population owning less than 10 hectares each own 89.95% of the land, and the state of inequality prevalent is no larger very alarming.

Thus, the division into classes through the values of a quantitative variable cannot be made according to any fixed rule but has to be decided according to what we already know about the problem being studied and what sort of conclusions we are trying to reach. The general rule is that the class width should not be too small as that can result in having classes with zero frequencies, and should not be too large as they may smoothen out any significant differences present in t data. Usually, one should begin with fairly small widths and then redefine the classes with larger widths or even widths of unnequal size.

16.7 Relative Frequency Table

As explained in the text-book the use of relative frequencie mainly serves two purposes. One is to make it easier to understand the information provided by the frequency distribution. Th discussion on Page 311 of the relative frequency distribution given by Table 16.3 and 16.4 already describes this use of relative frequencies. The teacher can give other examples. What must be emphasised and made clar is that the kind of insight we have got through the relative frequency distribution we would not have obtained from the original frequency distribution. This may require framing suitable questions which cannot be quickly answered from looking at the frequency table but can be answered by using relative frequencies. One such question is as follows:

"If we assume that all persons, male and female, between the ages of 20 to 60 constitute the working force and the rest are considered as dependants how many dependants each member of the working force will have to support on the average?" The working force constitutes 43.4% (7.4+7.3+6.7+5.9+5.4+4.2+4.1+2.4) of the population. Hence, the remaining 56.7% (29.5+12.3+18.1 = 49.9% children and youth, 3.0+1.4+2.4=6.8% old people) are the dependants. Thus on the average $\frac{56.7}{43.4}=1.3$ dependants have to be supported by each working person.

The second main use of the relative frequencies is in comparing two frequency distributions as has been done through the example given in Table 16.6. That example was artifically constructed just to illustrate how relative frequencies enable us to compare two situations. Even in this artificial example students can be made to see that if relative frequencies are not known we would not be able to see easily if one village has a larger proportion of older people than the other. Many interesting relative frequency distribution can be obtained from the census table (Table 16.1). For each age-group we have a frequency distribution into two classes, rural and urban. Here the unit of observation is a person belonging to a particular age group and the variable of observation is the place of residenc with just two values; rural and urban. If we calculate the relative frequency for each of these distributions we have the interesting result that for most of the age-groups the percentage of persons living in rural areas is 86%. We can also obtain the relative frequency distributions of the rural and urban populations by age. If we do that we will find that compared

to the rural population the urban population has a slightly higher proportion of younger persons.

Lastly, the students must be taught to round off the percentage to either one or two decimal points when obtaining the relative frequency distributions and made to observe that the total of relative frequencies of the classes does not always add up to 100.

Finally, the statement made in the last para of Sec. 16.7 must be firmly established in the mind of the students.

16.8 Graphical Presentation of Frequency Distributions

The methods of drawing bar diagrams are already known to the students. The pie-diagram may be new. There is not much to explain in this section except that the methods, which have been illustrated for quantitative variables only, are equally applicable to qualitative variables also. Therein lies one of the differences between a bar diagram and a histogram. The bar diagram can be used for graphical presentation of any frequency distribution but the histogram can be used only for frequency distributions with classes based on the values of a continuous quantitative variable.

16.9 Measures of location and dispersion

The discussion which follows this section covers the important concepts of measures of location and dispersion. Their basic purpose is to give a summary representation of the frequency distribution based on the values of a quantitative variable of observation. Thus, apart from learning the methods of calculating them the student has to learn to make an intelligent use of the information supplied by them.

16.10 Measures of Location

(a) The arithmetic mean

The basic definition is given on p. 317. It is

$$x = \frac{x_1 + \quad \dots + x_n}{n}$$

and the fact that it has to be expressed not as a mere number but as $\frac{x_1 + \dots + x_n}{n}$ per unit of observation, as is explained on p. 317, must be emphasised. The next thing to be made clear

that this is the only definition of the arithmetic mean which basically stands for the sum of the values divided by the number of values. In many texts one finds the statement that $\Sigma f_i x_i / \Sigma f_i$ (discussed on page 318) is another definition of the mean, which is incorrect. The latter is a formula for calculating the arithmetic mean in a particular case and is actually derived from the original definition (as described on p 318). Finally, in the case of grouped data it must be emphasised that what we are getting is an approximate value of the arithmetic mean and not its exact value. The fact that this method of approximation, viz., replacing every value in a class by the middle value of that class, is a reasonabdle one, can be brought home to the students by giving them some numerical data and asking them to calculate the arithmetic mean first without grouping the data and then after grouping it. Then, they can be told the logic behind the method. When all values are replaced by the mid value then some are increased and some are decreased and we hope that the two types of error will partly cancel each other so that the approximate value is not much different from the exact value.

Problem 2, 3, 4 and 5 of Exercise 16.1 are the usual results on the effect on the arithmetic mean of a linear transformation of a variable x to ax+b. It was earlier used to devise a short cut to calculating the mean. However, that short cut need no longer be discussed as the advent of electronic computers has rendered it redundant.

Problem 6 is rather important. It is not merely an exercise in calculating the arithmetic mean of a group from the means of its sub-groups through the usual formula of the type

$$x = \frac{n_1\bar{x}_1+n_2\bar{x}_2+\ldots+n_k\bar{x}_k}{n_1+n_2+\ldots+n_k}$$

It is really supposed to bring home to the student that though the class-wise mean ages per student are less in school A. the mean age per student for the whole school is greater in School A as compared to School B. This happens because the relative frequency for higher age-class is more in School A as compared to School B and because of that the average age for the whole school increases.

(b) Median:

The median is another measure of location Like the arithmatic mean it too represents the centre of all the observations of a quantitative variable but in a sense different from the arithmetic mean. This difference has to be emphasised and kept in mind while interpreting the information provided by the median of a set of observations.

A new definition of the median has been provided in this book on p. 322. It is different from the usual one based on even and odd numbers of observations. This has been done because this is the definition the students will come across when they study the subject in the higher classes. Also, because the earlier definitions in terms of even and odd numbers of observations are really speaking not definitions but methods of calculating the median in different cases. These are accordingly discussed under the sub heading "Calculation of the Median" on p. 323-324. The teacher should try to make the students see that these rules to calculate the median follow from the definition of the median given on p. 322.

When we come to the calculation of the median from frequency tables we have to distinguish between two cases. In the first, the classes of the frequency table are defined through single values of the variable of observation. So, it is not really different from the earlier discussion except that repeated values have been put together in a class with its frequency. Hence the rule for calculating the median does not change. It just depends, as before, on whether the total frequency is an odd or even number.

When we come to a frequency table where the classes are defined by more than one value, in fact, represent a whole interval of values, then the problem is different. It is now a question of devising a method to obtain an approximate value of the median. This fact must be clearly brought home to the student and the expression.

$$M = l + \frac{n/2 - f_a}{f_1} \cdot h$$

on p. 326 shown to be the approximation. It should be pointed out, in particular, that because we are approximating, we do not make any distinction now between even and odd number of values, but use n/2 in all cases.

In problem 1 & 2 of Exercise 16.2 the important aspect is not the calculation of the median but the interpretation of its value.

In problem 3 the situation is different. Here the approximation method is to be used Total frequency n=357 so that n/2= 178.5 Thus. 20-24 is the median class. Now we can use the formula

$$M = l + \frac{n/2 - f_0}{f_1} h$$

Here $l = 20$ $n/2=178.5$, $f_0=53$, $f_1=140$, $h=5$ (Note that we cannot take $h=4$ because the class 20-24 really represents age twenty or more but less than twenty five years). Hence

$$M = 20 + \frac{178.5 - 53}{140} \cdot 5$$

$$= 20 + 4.48 \simeq 24.5 \text{ years}$$

Problem 4 is intended to show that in many cases the median is a more useful summary than the mean. Though the average size of a holding is 1.25 hectares the median shows that nearly half the peasants have holdings below 0.58 hectares. It would be a good exercise to ask the students to find the number of persons whose holding is less than the average i.e. 1.25 hectares. From Table 16.8 on page 319 we find this number to be 7116591 + 298638 + part of 2591431 The last term has to be approximated. If we follow the same method of approximation as we did for calculating the median from grouped data, we will get the last term equal to

$$\frac{1.25 - 1}{2-1} \times 2591431 \text{ (approx.)}$$

Thus, the number of persons with holdings less than the average in size may be taken as

$$7116591 + 2986638 + 647858$$
$$= 10751087$$
$$= 71\%$$

We now see that more than two-third of all persons have holdings less than the average in size.

The important idea to emphasise again at this stage is that the mean and the median are only summaries and hence can only convey a part of the total information. So, care has to be taken to come to any conclusion from the given values of the mean or the median.

16.11 Measures of Dispersion

This section introduces the student to the concept of dispersion and describes two standard measures of dispersion. These are the variance or its square root, and the mean deviation about the median.

In both cases it is shown that the measure of dispersion is built from the deviation of the observations x_i from the mean (for the variance) or the median (for the mean deviation), Starting with the deviation $x_i - \bar{x}$, or $x_i - M$ one tries to derive a single number which will somehow measure if these deviations are small or large in magnitude.

4 For the calculation of the variance one uses the identity

$$(x_i - \bar{x})^2 = \Sigma x_i^2 - \frac{(\Sigma x_i)^2}{x}$$

It must be emphasised that the method of calculating separately each deviation $(x_i - \bar{x})$ and then squaring it must never be used. Instead, one should always use the method of squaring the observations and then using the formula $\Sigma x_i^2 - \frac{(\Sigma x_i)^2}{x}$. I have seen many text books where the method of squaring the deviations is still followed in the solved examples. Not only should this be discouraged but the students should also be made to see why

the method of squaring the deviations should be replaced by $\sum x_i^2 - \frac{(\sum x_i)^2}{x}$. The first advantage is in the reduction of the number of arithmetic operations where n subtractions are replaced by only one subtraction and that at the cost of increasing the number of multiplications by one. The second advantage is that the chance of making calculation errors is reduced, since the chance of making an error is more while doing subtractions.

The student should also be told that while algebraically the three expressions

$$\sum(x_i-\bar{x})^2, \sum x_i^2 - n\bar{x}^2, \sum x_i^2 \frac{(\sum x_i)^2}{x},$$

are equivalent they can in practice, lead to different results because of rounding off errors. For example, take $\sum x_i=1151$, $n=39$, Then $\bar{x} = \frac{1151}{39} = 29.51282$. We will normally round it off to 29.5 which will give $n\bar{x}^2 = 39 \times 29.5 \times 29.5 = 33939.75$. But $\frac{(\sum x_i)^2}{n} = \frac{(1151)^2}{39} = 33969.256$. If we round off $\bar{x}$ to two decimals we will take $\bar{x} = 29.51$ giving $n\bar{x}^2=33962.763$ which is till slightly different from $(\sum x_i)^2/n$. If we round off $\bar{x}$ to 3 decimal places, taking $\bar{x}=29.513$, we got $n\bar{x}^2=33969.669$. Therefore, one should always compute $n\bar{x}^2$ as $(\sum x_i)^2/n$ even though algebraically they are equivalent.

The short cut method described on pages 335 336 is based on the effect on variance of a linear transformation $u = \frac{x-A}{n}$ of the variable x. With the use of modern electronic calculators it is likely that the calculation may lead to numbers beyond the display capacity (usual 8 digits) of small pocket calculators. In such cases the use of the transformation $u = \frac{x-A}{n}$ whereby the observations x_i are replaced by observation u_i which are smaller

in magnitude is advisable. The maximum advantage of this method occurs when we have data grouped in equal size intervals. In such cases, we can by a suitable choice of A and h make the u_i values equal to 0 ± 1, ± 2, ± 3,..... thus making the computation of the variance extremely easy. This is what happens in the example on p. 336. But, the student should be made to see that such a simplification was made possible becauses A was taken to be as one of the mid-values and h was taken equal to the gap between the mid-values of the intervals which are all of the same width.

5. CHAPTER TESTS (ORAL)

1. What is the difference between a census and a survey?

2. What is the purpose of calculating the "Cost of living index"?

3. What is meant by the "age-distribution of a population"?

4. What is "Sex-ratio" : can it have a value (a) equal to 1, (b) less than 1, (c) more than 1 ?

5. What is the meaning of "age-specific death rate" ?

6. What are the units of observation and variables of observation in the following

   (a) Table 16.1 (b) Table 16.7

   (c) Problem 4(b) of Exercise 16.1 (d) Problem 1 of Exercise 16.2

   Which variables are qualitative and which quantitative ?

7. What is meant by the "relative frequency" of a class ?

8. If a constant is added to (subtracted from) all the values of a quantitative variable what happens to the values of the mean, median, variance, standard deviation, mean deviation about the median ?

9. If every value of a quantitative variable is multiplied (divided) by a constant what happens to the values of the mean, median, variance, standard deviation, mean deviation about the median ?

10. The frequency distribution of the inhabitants of village A by age gave the mean age as 25.6 years per inhabitant and standard deviation as 3.25 years. For village B the mean was 25.7 years and standard deviation 8.34 years. What do you

11. The median income of the workers in a factory was Rs.780 p.m. with a mean deviation of Rs. 20/-. For the persons working in the office in the same factory the median income was Rs. 1215/- p.m., with a mean deviation of Rs. 85/- what conclusions can you draw from this information ?

ANSWERS

1. See Page 300
2. Mainly used to see the effect of prices on the living standards and to decide if the dearness allowance is to be increased or decreased.
3. It tells us about the numbers (or percentages) of persons falling in different age-groups.
4. It is the number of females divided by the number of males in a given group. It can have values $\lesseqgtr 1$.
5. It is the death-rate of persons belonging to a given age-group.
6. (a) Persons alive at the time of the Census (Unit); Age (Quantitative variable); sex, place of residence (qualitative variables)

   (b) families (unit) : number of children (quantitative variable)

   (c) Letters posted on a given day (Unit) : amount of postage (quantitative variable):

(d) Day on which attendance recorded (Unit); No. of students absent (quantitative variable).

7. It is the frequency of the given class divided by the total frequency.

8. The same constant is added to (subtracted from) the mean and the median; no change in variance, standard deviation and mean deviation about the median.

9. The mean, median, standard deviation and mean deviation are multiplied (divided) by the same constant, the variance is multiplied (divided) by the square of the constant

10. Though the mean ages are almost equal for the two villages, village A has a larger proportion of inhabitants, as compared to village B, with ages near to the mean age.

11. Most of the workers have almost the same income but the incomes of office staff show greater differences. Also, a large majority of office staff has higher income as compared to the workers.

CHAPTER - 17

LINEAR PROGRAMMING

1. Introduction:

Systems of simulataneous equations and their solutions were discussed in earlier classes. But the world of equations is highly restrictive and we deal here with only inequalities (inequations).

Let us consider the problem faced by a housewife in her daily routine. She wants to find the daily diet satisfying the minimum daily requirements of nutrients by spending minimum on different food articles. Similar is the situation for a manager who wants to make the best use of his limited resources, for producing different items to meet the demand and maximize his profit. In these situations. We implicitly deal with inequalities rather than equalities. In this chapter we deal with inequalities to tackle the problems similar to these mentioned above

The algebra of inequalities was studied in detail by the Russian mathematician and economist L.Kontarovitch after 1939. In 1947, George B. Dantzig discovered a simple algorithm called the simplex method to solve problems involving linear inequalities along with linear objective function which is to be maximized or minimized. With the advent of computers, very large problems have been solved using this method.

The purpose of this chapter is to study a special class of problems known as linear programming problems. The problems involving linear inequalities along with linear objective function which is to be maximized or minimized are known as

linear programming problems. When the objective function is nonlinear and the constraints are also non linear inequalities then the problem is known as nonlinear programming problem. However, in this chapter we will be dealing with only linear programming problems.

Operations Research has been described as the scientific approach to decision making problems. This new science came into existance during World War-II. Here, we will be concerned with particular problems of Operations Research called the linear programming problems. The subject dealing with linear programming problem is called linear programming.

Linear programming has found wide variety of applications, particularly, in defence, economics and airline industry etc. In the field of telecommunication, large network, problems have been solved using linear programming methods. Further the study of algorithmic analysis has developed very rapidly after the advent of linear programming.

2 CONTENT ANALYSIS

The following terms and concepts occur in this chapter:

i) The equation $ax+by+c=0$, where a and b are not simultenaously zero, is called a linear equation in two variables x and y. Similarly the equation $a_1x_1+a_2x_2+\ldots\ldots a_nx_n+k=0$, where $a_1, a_2 \ldots, a_n$ are not simulteneously zero, is called a linear equation in n variables $x_1, x_2 \ldots, x_n$.

ii) The linear equation $ax+by+c=0$ represents a straight line.

iii) Every point (x,y) on the straight line $ax+by+c=0$ is a solution of the equation $ax+by+c=0$. Therefore, the solution set of $ax+by+c=0$ is an infinite set.

iv) $ax+by+c < 0$ is called a linear inequation intwo variables. Similarly $ax+by+c > 0$ is also a linear inequation in two variables.

v) The linear inequation $ax+by+c < 0$ represents a half-plane and $ax+by+c > 0$ represents another half-plane with respect to the line $ax+by+c=0$.

vi) $ax+by+c \leq 0$ is a compound statement with the connector "or". The solution set of $ax+by+c \leq 0$ is the union of the solution sets of $ax+by+c < 0$ and $ax+by+c=0$.

vii) $\begin{cases} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{cases}$

is a compound statement where the simple statements $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$ are connected by "and". This is called a system of equations.

viii) The solution set of a system of equations is the intersection of the solution sets of its constituents

ix) We can have a system where the constituents are equalities or inequalities or both. Mostly, we are dealing with such systems only.

x) The solution set of a system of inequalities may be an empty set or a finite set or an infinite set.

xi) A problem of the type

Optimise (maximise or minimise)

$C(x,y) = c_1x+c_2y$

subject to the constraints

$$\left.\begin{array}{l} x \geqslant 0 \\ y \geqslant 0 \end{array}\right\} \qquad (1)$$

and

$$\left.\begin{array}{l} a_1x+b_1y \leqslant c_1 \\ a_2x+b_2y \leqslant c_2 \\ \text{-----------} \\ a_nx + b_ny \leqslant C_n \end{array}\right\} \qquad (2)$$

is called a linear programming problem. $C(x,y)$ is called the objective function of the problem. (2) is called the linear constraints and (1) is particularly called the non-negativity constraints.

xii) A region is said to be a convex region if the line segment joining any two points of the region lies entirely in the region.

xiii) The region representing the solution set of the constraints is called the region of feasible solution.

xiv) A feasible solution which also optimizes the objective function is called on optimal solution.

xv) Since the set of feasible solutions to a linear programming problems is a convex set, the proof of this statement is beyond the scope of this chapter, therefore, the optimal solution lies at a vertex of the convex polygon.

xvi) It is clear that any solution (x,y) of a linear programming problem which satisfies the condition $x \geqslant 0$, $y \geqslant 0$ lies in the first quadrant only. This, the search for a desired solution (x,y) will be restricted to the first quadrant only.

3. LEARNING OUTCOMES

a) Essential learning outcomes for all

i) Use of linear inequalities to describe certain situations and to obtain their solution set. To realise that the solution set may be bounded or unbounded, may consist of a single solution, or more than one solution or may not have a solution at all. Since the solutions are represented by points in a plane the student must understand what is meant by the solution set being bounded or unbounded, and what is meant by the solution being unique in the case of subsets of a plane. [illegible] He should also be able to see that if there is more than one solution, then the number of solutions is infinite because of the convexity of the solution set. Due to convexity, if (x_1,y_1) and (x_2,y_2) are two solutions, then every point on the line segment joining these two solution points is also a solution.

ii) Learning the special form in which a linear programming problem is mathematically presented, specially the non-negativity condition, the linearity of the function to be minimised or maximised, and the fact that all

inequalities describing the constraints, other than the non-negativity constraints, are written as $\leq$ inequalities.

iii) Learning to formulate practical problems mathematically as a linear programming problem.

iv) Using the graphical method to obtain the solution to a linear programming problems in two variables.

b) <u>Learning outcomes for the higher groups:</u>

Since the topic has been treated at an elementary level restricted to the use of the graphical method involving two variables only there are no immediate learning outcomes for the higher groups. However, the following learning outcomes are possible for such groups if the teacher, and the good students have the time to interact to go a bit into the theoretical foundations.

i) To visualise that there can be simple problems involving more than two variables which can also be treated as linear programming problems.

ii) To see that by the use of vector and matrix notation a linear programming problem can be presented in the following general form which takes care of any number of variables that may be presented:

Maximise (or minimise) $c_1x_1+c_2x_2+\ldots+c_nx_n$, subject to

$$\underset{\sim}{A}\,\underset{\sim}{x} \leq \underset{\sim}{b} \text{ (constraints)}$$

$$\underset{\sim}{x} \geq \underset{\sim}{0} \text{ (non-negativity)}$$

iii) To understand that instead of using differential calculus methods to obtain the maximum (or minimum) value we are able to use a simpler method because of the special nature of the problem in which the function to be maximised or minimised is linear and the solution set is a convex polygon.

4 TEACHING STRATEGIES

In this section the number of each subsection is in accordance with the text books.

Motivation for the development of Concepts

17.1 Linear Constraints

Teachers are familiar with equations and their solution set There are many situations in our day to day life which can be modelled in the form of equalities and inequalities. Many such situations are explained in the textbook.

A set of equations may have a unique solution or may not have a solution or may have an infinite number of solutions. Similarly, the set of inequalities may have a unique solution or may not have a solution or may have an infinite number of solutions.

Teachers can clarify the above mentioned concepts as follows:

In the case of equations, the system of equations:

$2x + 5 = 7$

$3x + 7 = 10$

has a unique solution $x = 1$. The following system of

equations

$$3x = 4$$
$$2x = 7$$

has an empty solution set since no value of x satisfies both the equations simultaneously. On the other hand the system of equations

$$3x = 2x + x$$
$$x + 5 = 5 + x$$

has an infinite solution set.

Similarly equations in two variables can be taken to clear the above concepts.

It can be easily seen that the system of inequalities

$$\left.\begin{array}{l} 2x \leqslant 2 \\ 3x \geqslant 3 \end{array}\right\}$$

has the solution set $\{1\}$ which is unique. But the system of inequalities

$$\left.\begin{array}{l} 2x \leqslant 2 \\ 3x \geqslant -3 \end{array}\right\}$$

has solution set $\{x \mid -1 \leq x \leq 1$ which is an infinite solution set. On the other hand the following system of inequalities:

$$2x \geqslant 2$$
$$3x \leqslant -3$$

has an empty solution set as no value of x satisfies both the inequalities simultaneously.

An important property of linear constraints of two variables x and y is that the set of points (x,y), i.e. the solution set, for which all the inequalities are satisfied is either empty. or is bounded by straight lines, i.e. bounded region is a polygon, or is an unbounded region with straight line boundaries. This can be made clear by the following examples:

The thick lines represent the fact that the points of the solution set lie on the lines.

Example 1: Obtain the solution of the system

$$x - y \leqslant -1$$
$$-x + y \leqslant 0$$
$$x \geqslant 0, \quad y \geqslant 0$$

Solution: To determine the solution of the given system, we draw the graph of the each of the inequalities of the system, on the same coordinate plane. The graph of the system is shown in Fig. 17.1

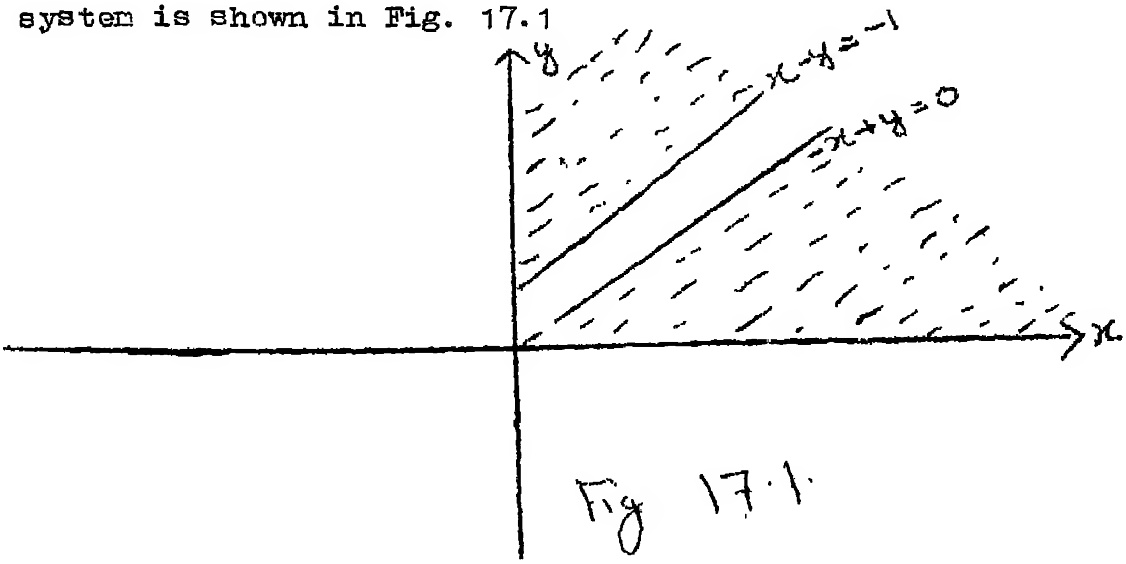


Fig 17.1

It is clear that there is no point which satisfied both the inequalities simultaneously. Thus, the solution set is empty.

<u>Example 2</u>: Obtain the solution of the system

$$2x - y \leqslant 2$$
$$x \leqslant 3$$
$$x \geqslant 0, \quad y \geqslant 0$$

<u>Solution</u>: To find the solution of the system, we draw the graph of the system in Fig. 17.2. The shaded region of the figure represents to the solution set. Some of the common points of the solution set are (0,0), (2,2), (2,3) and so on. In fact, all the points in the shaded region

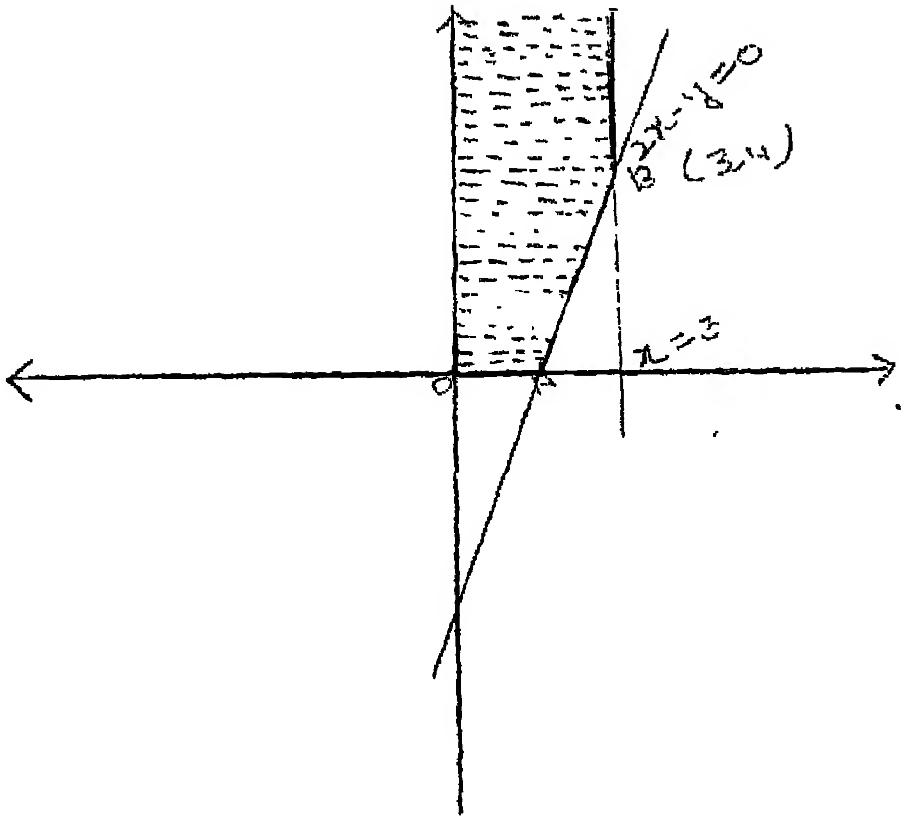


Fig. 17.2

are the points of the solution set. Here the solution set is an <u>unbounded region</u>.

<u>Example 3</u>: Solve the system of inequalities

$$x + y \leq 4$$
$$3x + 8y \leq 24$$
$$10x + 3y \leq 30$$
$$x \geq 0, \quad y \geq 0$$

<u>Solution</u>: The graph of the system is shown in Fig. 17.3

The shaded region of the figure along with the boundary points is the solution set of the system. Any point within the region and lying on the boundaries satisfied the inequalities. The bounded region is the solution set.

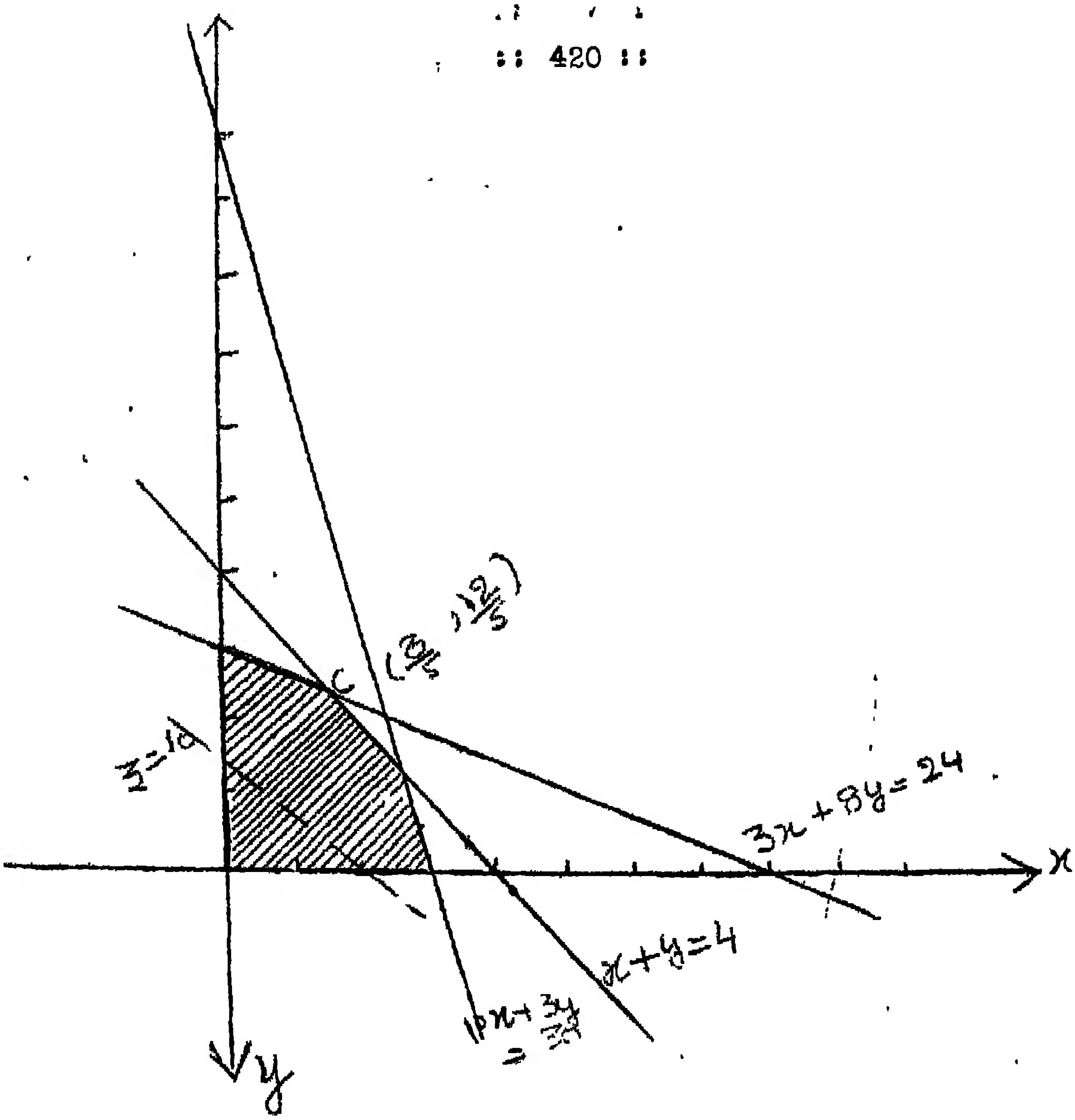


Fig. 17.3

Till now we have studied a system consisting of inequalities only. In the following examples, we shall now be studying the system consisting of one equation and one inequation. If the system consists of several inequations a an equation, then the method of finding the solution set is the same as in the following examples.

Examples 4:

Solve the following system

$$\left.\begin{aligned} y &= 2x \\ x &< 0 \end{aligned}\right\}$$

Solution: The graph of the system is shown in Fig. 17.4

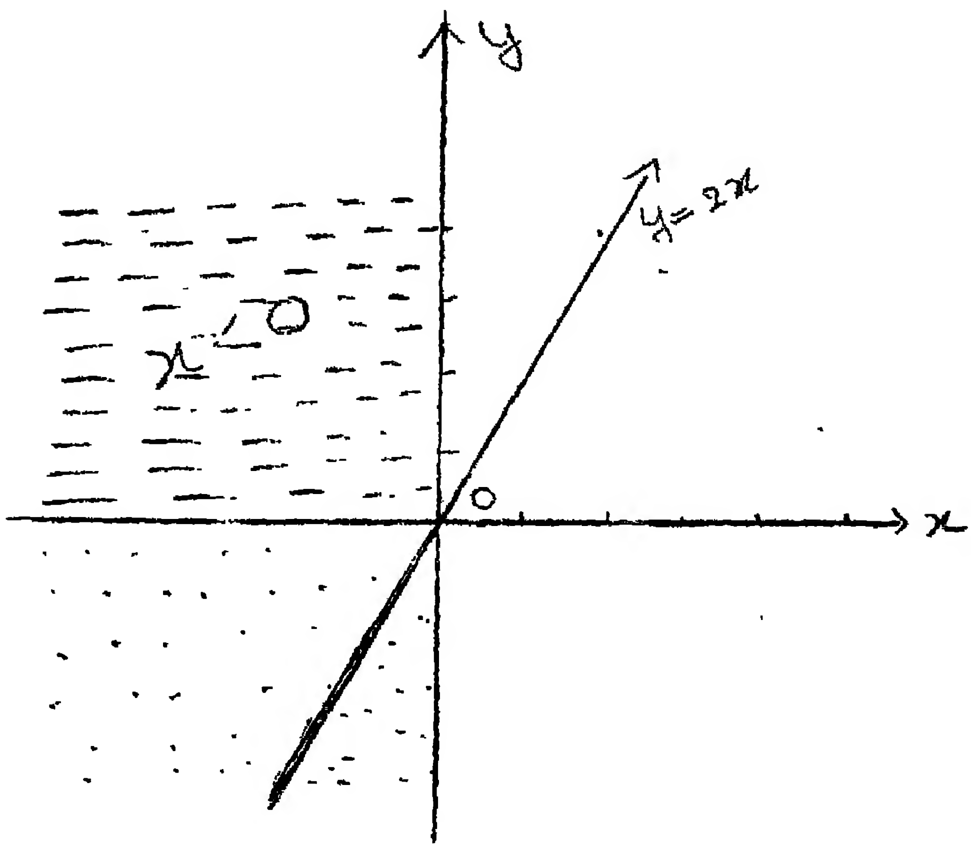


Fig. 17.4

The thick portion of the line y=2x not including the point (0,0) is the graph of the solution set. Thus, the thick line is the required solution set of the system.

<u>Example 5:</u> Solve the system

$x - 2y = 2$

$x + 2y \geqslant 6$

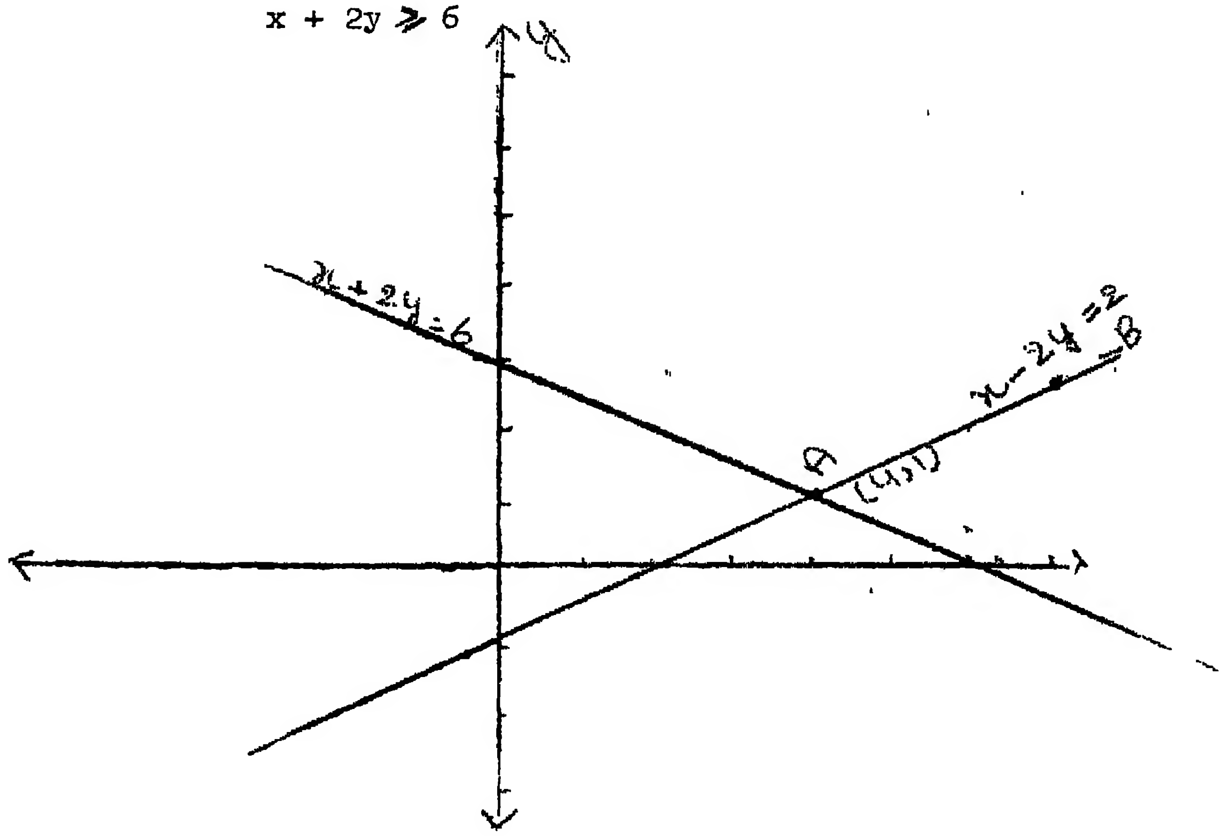


Fig. 17.5

Solution.

From the graph of the system it is clear that the thick line AB including the point A(4,1) is the graph of the solution set. However, we can verify the solution algebraically also as is done below.

By Substituting x for y in $x+2y \geqslant 6$, we get $2x \geqslant 8 \Rightarrow x \geqslant 4$.

Hence, the solution set is

$$\{(x,y) \mid x-2y=2 \text{ and } x \geqslant 4.\}$$

Thus, the ray AB is the required solution set of the system.

Note that the set of pairs of values (x,y) which satisfyes all the inequalities or equalities of the system is called the set of feasible solutions of the problem.

17.2 LINEAR PROGRAMMING

Due to the extensive use of computers, all the teachers, are familiar with the term 'programming'. Here, by programming, we mean the set of steps to be considered for formulating the everyday life problems into mathematical language and solving the same. As seen above, such situations give rise to linear equalities or linear inequalities, therefore, the term linear programming. By a linear programming problems, we mean a problem where our aim is to optimise a linear objective function subject to given linear constraints.

Below, we mention a linear programming problems.

A firm can produce three types of cloth, say, A, B and C. Three kinds of whool are required for it, say red wool, green wool and blue wool. One unit of length of types A cloth needs 2 yeards of red wool and 3 yards of blue wool, one units of length of type B cloth needs 3 yeards of red wool, 2 yards of green wool and 2 yards of blue wool and one unit of type C cloth needs 5 yards of green wool and 4 yards of blue wool. The firm has only a stock of 8 yards of red wool, to yards of

green wool and 15 yards of Blue wool. It is assumed that income obtained from one unit of length of type A cloth is Rs. 3, of type B cloth is Rs. 5 and of type C cloth is Rs. 4 and it is to be maximised. The mathematical formulation of the problems is given below.

Suppose the firm produces x_1, x_2, x_3 units of A, B and C types of cloth respectively. Then the total profit (in rupees) is given by

$$z = 3x_1 + 5x_2 + 4x_3$$

The total units of red wool required to produce x_1 units of type A and x_2 units of type B cloth are given by

$$2x_1 + 3x_2,$$

and the total units of green wool and blue wool required to produce x_2 units of type B; x_3 units of type C and x_1 units of type A, x_2 units of type B and x_3 units of type C cloth are given by

$$2x_2 + 5x_3$$

$$\text{and} \quad 3x_1 + 2x_2 + 4x_3$$

respectively. Since the firm has only 8 yards of red wool, 10 yards of green wool and 15 yards of blue wool, therefore, we must have

$$2x_1 + 3x_2 \leqslant 8$$

$$2x_2 + 5x_3 \leqslant 10$$

$$3x_1 + 2x_2 + 4x_3 \leqslant 15$$

Also since it is not possible for the firm to produce negative units of different cloth, it is obvious that we must also have

$$x_1 \geqslant 0, \quad x_2 \geqslant 0 \text{ and } x_3 \geqslant 0.$$

This is called the non-negativity condition.

Hence, the firm's production problem can be put in the following mathematical form:

Find x_1, x_2, x_3 such that the total profit (objective function)

$z = 3x_1 + 5x_2 + 4x_3$

is maximised subject to the following constraints

$$2x_1 + 3x_2 \leqslant 8$$
$$2x_2 + 5x_3 \leqslant 10$$
$$3x_1 + 2x_2 + 4x_3 \leqslant 15$$
$$x_1 \geqslant 0 \quad x_2 \geqslant 0, \quad x_3 \geqslant 0$$

The teachers should take some more problems and also the students to write them in mathematical form.

As you know that the set of values x, y which satisfy the constraints and the non-negativity condition is called a feasible solution . A feasible solution which also optimizes the objective functuion is called an optimal solution.

17.3 SOLUTION OF A LINEAR PROGRAMMING PROBLEM

To obtain the solution of the problems involving inequalities using graphical method it can be observed that the optimal value of the objective function is obtained at the boundary of the region of the feasible solution by moving the straight line which represents the objective function in that region. This is possible due to the special nature of the problem in which the function to be maximised or minimised is linear and the solution set is a convex polygon. This property of the function gives us a simpler method to solve the problem instead of differential calculus methods. Through the following examples we see that the optional solution of a linear programming is obtained at the boundary of the region of the feasible solution.

Let us see the following linear programming problem:

Example 6: Maximize $z = 5x + 7y$

subject to: $x+y \leqslant 4$

$$3x + 8y \leqslant 24$$

$$10x + 3y \leqslant 30$$

$$x \geqslant 0, \quad y \geqslant 0$$

Solution: Please refer to Fig. 17.3 of example 3. The shaded region in the figure is the feasible region bounded by the darklines. Note that every point within this region and on the boundary of the region satisfies all the constraints. Now, our aim is to find a point in this region including boundary which gives the maximum value of z. In order to find this point we draw a line corresponding to some arbitrary numerical

value of z and move this line above and parallel to itself, until it contains a point of the feasible region. By doing so, we find that the required point is C $(\frac{8}{5}, \frac{12}{5})$ and the maximum value of z is $\frac{124}{5}$. Note that the point C is the intersection point of the lines whose equations are $x+y=4$ and $3x+8y=24$

Note that we can draw further lines corresponding to the greater values of z, but those lines will not contain even a single point of the feasible region.

Note that in most linear programming problems the set of feasible solution is a polygon in the positive quadrant, that is, a closed figure bounded by straight lines. The set of feasible solution is also a convex set which means that if any two points of the set of feasible solution are joined by a straight line, the straight line lies completely in the set of feasible solution. In Fig. 17.6, polygon at (i) and (iii) are convex while

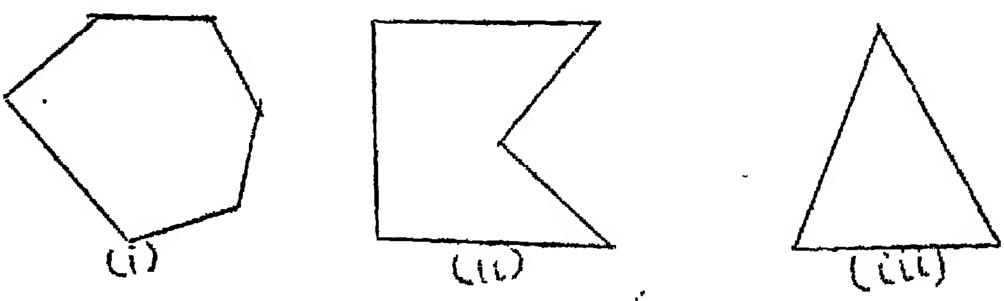


Fig. 17.6

that at (ii) is not convex.

The fact that the feasible region is a closed set is important for the solution procedure of a linear programming problem. We want to either maximize or minimize our objective

function which is linear. It is an important result the maximum or minimum of a linear objective function over a closed convex set if it exists, is attained at a point (points) on the boundary.

With reference to the Example 6 we note that the value of the objective function is $z=0$ at the point $(0,0)$. At the point $(0,3)$, $z=21$, at $(4,0)$, $z=20$ and at the point $(\frac{8}{5}, \frac{12}{5})$, $z=\frac{124}{5}$ which is the maximum value of z. The point $(\frac{8}{5}, \frac{12}{5})$ is the boundary point of the feasible solution set. Infact, it is a point of the convex polygon of the feasible region.

When the feasible region is bounded it is easy to find the optimum solution. What can we say about the optimum solution when the feasible region is unbounded ? Let us refer to example 2.

Example 7: Maximize $z=6x-2y$

$$\text{s.t. } 2x-y \leqslant 2$$
$$x \leqslant 3$$
$$x \geqslant 0, \quad y \geqslant 0$$

Solution: Note that the feasible region as shown in Fig. 17.2 is unbounded but the finite maximum exists. Thus always bounded region does not mean unbounded solution. The optimum solution is the point $B=(3,4)$ and the maximum value of $z=10$.

It is important to note that with the unbounded region the optimal solution may exist. But, on the other hand, if the feasible solution set is empty, i.e. there is no point which satisfies all the constraints, then there is no optimal

solution of the given linear programming problem (l.p.p)

<u>Example 8:</u> Solve the l.p.p. where we have to maximize

$Z = x+y$

subject to the constraints

$x + y \leqslant 2$

$-4x + y \geqslant 4$

$x \geqslant 0 \quad y \geqslant 0$

Solution: As shown in Fig. 17.7 the two half planes

$$x + y \leq 2 \text{ and } -4x + y \geqslant 4$$

do not intersect and thus have no point in common.

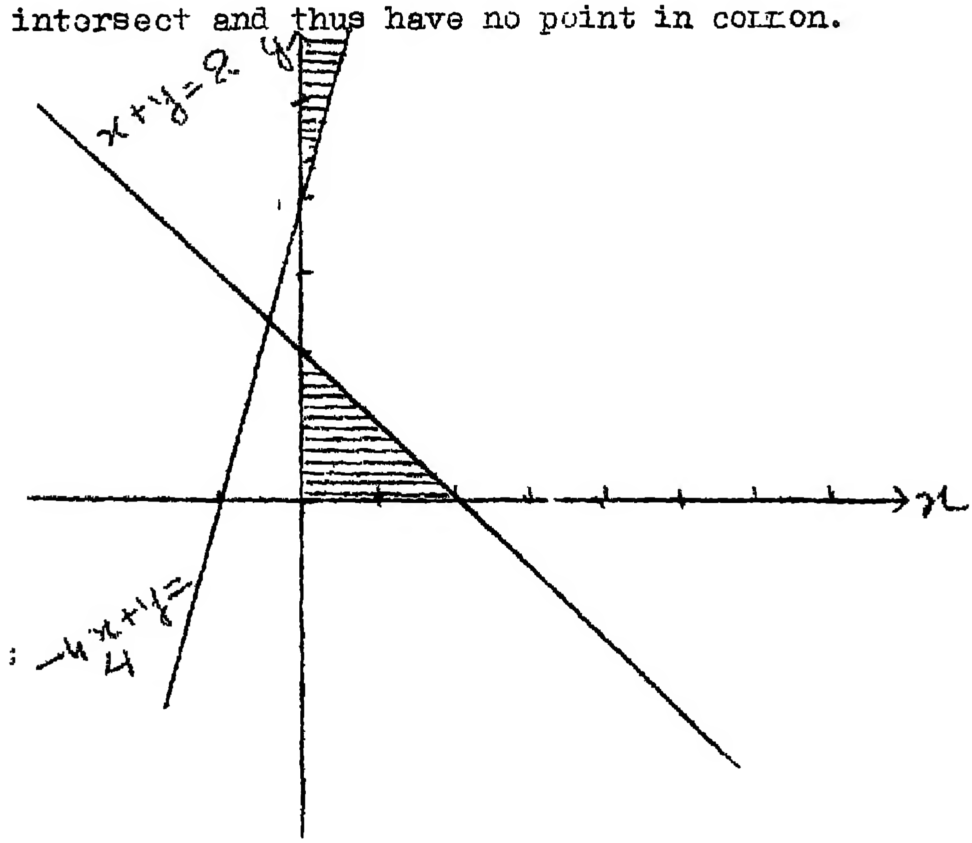


Fig. 17.7

As there is no point (x,y) which can lie in both the regions, there exists no solution to the given l.p.p.

So far, we have dealt with l.p.p. having bounded feasible region with a unique optimal solution and unbounded feasible region with a unique optimal solution. Below we solve a l.p.p. having a bounded feasible region but there exists an infinite number of optimal solution to the given l.p.p.

<u>Exercise 9</u>: Solve the l.p.p. in which the objective function

$$Z = x + y$$

is to be maximised subject to the constraints:

$-2x + y \leq 1$

$x \leq 2$

$x + y \leq 3$

$x \geq 0 \quad y \geq 0$

<u>Solution</u>: The shaded region of Fig. 17.8 represents the feasible region of the given l.p.p. and we observe that the region is bounded. As the

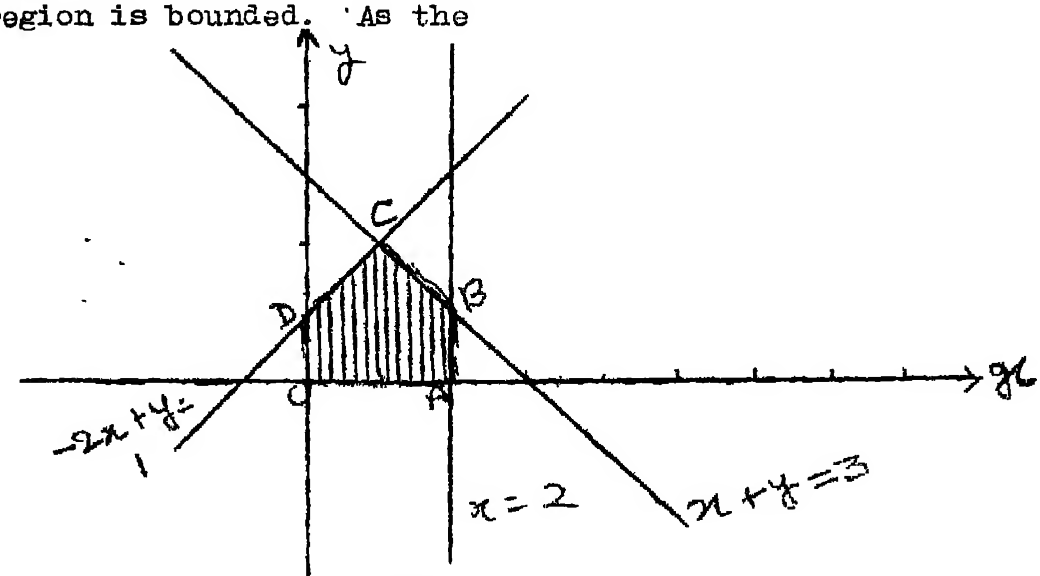


Fig. 17.8

value of the objective function lies at the points of a convex polygon. Therefor... the values of z at the four points.

$O=(0,0)$, $A=(2,0)$, $B=(2,1)$, $C=(0,7,2,3)$ and $D=(0,1)$.

are given by

$Z(O)=0$, $Z(A)=2$, $Z(B)=3$, $Z(C)=3$, $Z(D)=1$.

Obviously, the maximum value of Z occurs at two points B and C of the shaded convex region OABCD. It is important to note that all those points lying on the line BC joining the points B and C will give the maximum value. Hence, there exists an infinite number of optimal solution to the given l.p p

In the following l.p.p the feasible region is unbounded Do [illegible] re exists an optimal solution to the given l.p.p.?

Exercise: 40: Solve the l.p.p. where $Z=x+y$ is to be maximized subject to the constraints:.

$$2x - y \leq 0$$
$$x \leq 3$$
$$x \geq 0, \quad y \geq 0$$

Solution: Refer to Fig. 17.2 The feasible region is unbounded The value of the function Z at point $B=(3,4)$ is 7 which is maximum with respect to the points O and A. Note that the ray lies in the feasible-region and value of the function keeps on increasing on this live. Hence, Z can be made arbitrarily large, and the problem has no finite maximum value of Z. In such a case we say that the problem has an unbounded solution

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Note: We do not expect any linear programming problems representing some practical situation to have an unbounded solution, since this would imply, for example, the feasibility of an infinite profit. However, the limitation of resources and the impossibility of making arbitrarily large profits are precisely the reasons for our interest in using linear programming. Nonetheless, it occasionally happens that a mistake in the formulation of an actual problem leads to an unbounded solution.

(b) Explanation of examples from the textbook

For the exercises of this section we refer to the Mathematics book for class X, Part-I, which was published in March, 1989.

1 This has reference to Fig. 17.5 in the textbook. Our region is defined by the following inequalities:

$$2x + 3y \leq 35$$

$$x \geq 2$$

$$y \geq 3$$

Therefore the shaded region in Fig. 17.9 given below is the required region. The shaded region

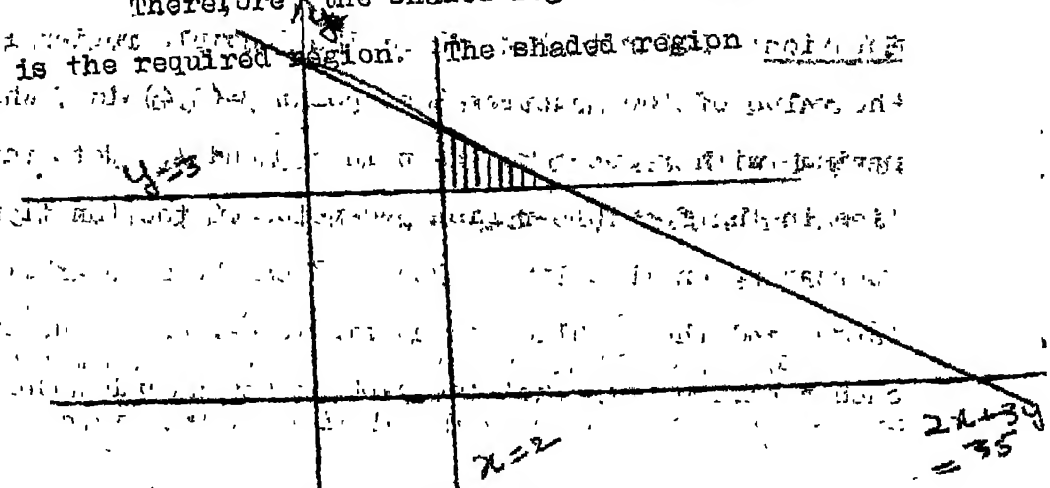

Fig. 17.9

as printed in Fig. 17 5 of the text book is incorrect.

2. The Fig 17.15 in the textbook is incorrect, the correct figure is given below:

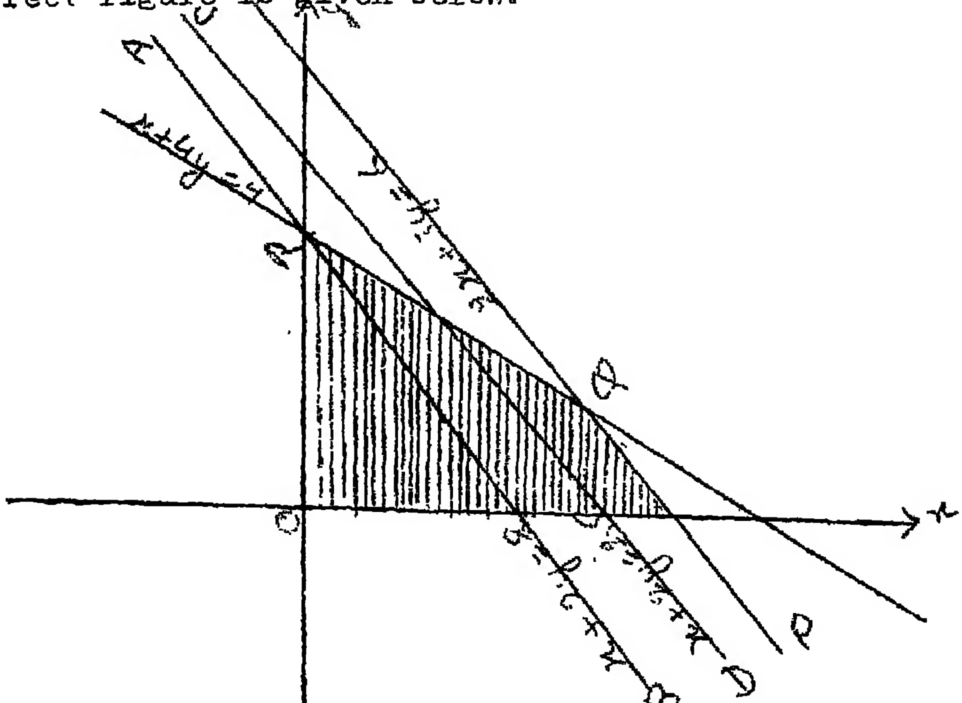

Fig. 17.10

3. The Fig. 17 16 as given in the textbook is not correct. The correct figure is given below:

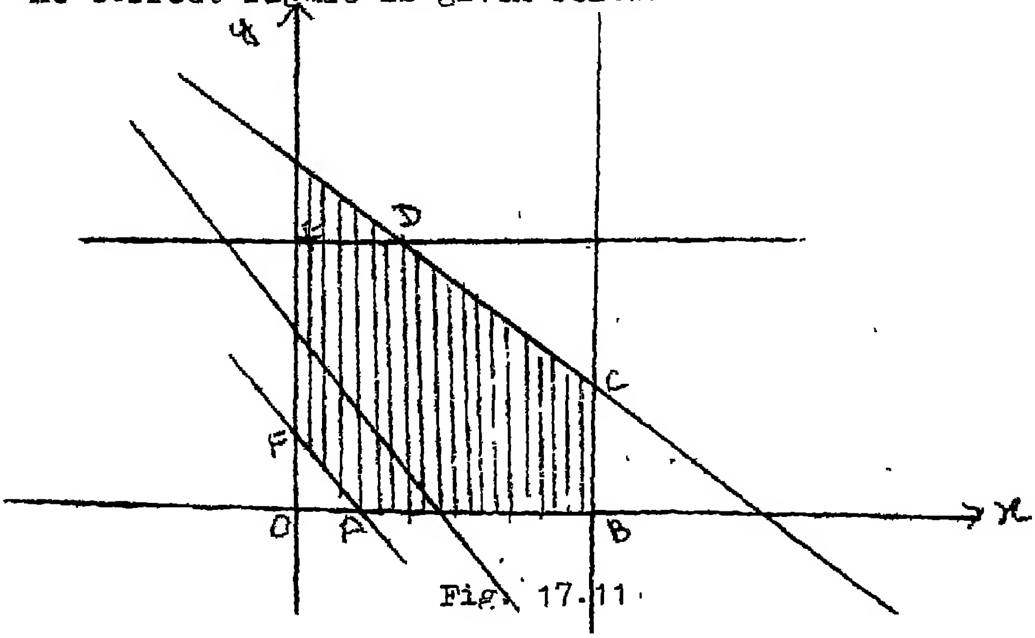

Fig. 17.11

5. ANSWERS AND SOLUTIONS TO SELECTED PROBLEMS IN EXERCISES 17.1 AND 17.2 OF THE TEXT BOOK

1. Answer to question 6 of exercise 17.1 is wrong. The correct answer is :

$2x + 3y \geq 3$

$x - 6y \leq 3$

$3x+4y \leq 18$

$-7x + 4y \leq 14$

$x \geq 0, \quad y \geq 0$

2. The linear programming problem of question No. 4 of exercise 17.2 is as follows:

Let x km denotes distance travelled at 25 km/h

y " " " " " 40 km/h

∴ Maximise x+y

subject to

$2x+5y \leq 100$

$\frac{x}{25} + \frac{y}{40} \leq 1$

$x \geq 0, \quad y \geq 0$

Max. = 30km. at $\left(\frac{50}{3}, \frac{40}{3}\right)$

6. CHAPTER TEST (ORAL/WRITTEN)

Answer the following as true or false:

1. The region between two concentric circles is convex (False)

2. Every plane triangular region is convex (True)

3. The feasible region of $x \geq 0, \quad x \leq 2, \quad y \geq 0, \quad y \leq 2$ is unbounded (False)

4. The solution set of $x \geqslant 2$, $x \leqslant 1$ is empty (True)

5 The feasible region of a linear programming problem is always nonempty. (False)

6. The feasible region of a linear programming problem is always bounded (False)

7. Every linear programming problem has a unique optimal solution. (False)

8. If the feasible region is bounded then for any linear objective function of a l.p.p. There is a finite optimal solution. (False)

9. If a linear programming problem have two points as optimal solutions then every point on the line segment joining those two points is also optimal having the same optimal objective function value. (True)

10. If exactly two points are optimal in a l.p.p. then they are adjacent points of the feasible region. (True

B. Solve the following linear programming problems:

11. Min. $Z=2x - 10y$

subject to the constraints

$x-y \geqslant 0$

$x-5y \geqslant -5$

$x \geqslant 0 \quad y \geqslant 0$

(min. Z=0 at (0,0)

12. min. $Z=2x+y$

subject to the constraints:

$x+2y \leqslant 10$

$x+y \geqslant 1$

$y \leqslant 4$

$x \geqslant 0, \quad y \geqslant 0$

(min $z=1$ at $(0,1)$)

13. max. $Z = -5y$

subject to the following constraints:

$x + y \leqslant 2$

$x + 5y \geqslant 10$

$x \geqslant 0, \; y \geqslant 0$

14 Max. $4x + 5y$

subject to the constraints:

$x + y \geqslant 1$

$-2x + y \leqslant 1$

$4x - 2y \leqslant 1$

$x, y \geqslant 0$

(Solution set is unbounded)

15. Max. $Z=4x+3y$

subject to the constraints:

$-2x+3y \leqslant 9$

$x-5y \geqslant -2$

$x \geqslant 0, \quad y \geqslant 0$

(There exists no solution)

16. Max. $A=x+y$

subject to the constraints

$-2x+y \leq 4$

$x - 3y \leq 6$

$x \geq 0,\ y \geq 0$

(Solution set is unbounded)

17. Minimize $Z=x-2y$

subject to the constraints

$x-y \leq 2$

$3x - y \geq -3$

$x \geq 0,\quad y \geq 0$

(Solution set is unbounded)

18. Max. $Z = -x+y$

subject to the constraints

$x+y \leq 3$

$x - y \geq 5$

$x \geq 0,\quad y \geq 0$

(No solution as feasible solution set is empty)

19. Max. $Z=x + 3y$

subject to the constraints

$5x + 2y \leq 7$

$3x + y \geq 4$

$x \geq 0,\quad y \geq 0$

20. Maximize $Z = 2x + 3y$

subject to the constraints

$x + y \leq 2$

$4x + 6y \leq 9$

$x \geq 0, \quad y \geq 0$

(Optimal solution is infinite)

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3. Harvey M Wagner, Principles of Operations Research with Applications to Managerial Decisions, Prentice-Hall of India Private Limited, 1974.

CHAPTER - 18

ALGORITHMS AND FLOWCHARTS

1. INTRODUCTION

The most significant scientific event in the history of twentieth century is the advent of computer. Computer revolution has taken place every where in the present day life. Industrial revolution helped to increase the muscle power of mankind. Computer revolution has helped to increase the brain power of mankind. Enormous amount of information is collected and analysed, with enormous speed, accuracy and efficiency by the use of computers, and then stored in the computer itself. Great mathematicians like Pascal, Leibnitz, Babbage, Turing and John Von Neuman have contributed a lot in this revolution. Computer is an electronic machine. How it is built and manufactured is not our concern in mathematics curriculum? Only those aspects of computer revolution which have bearing on mathematics and involve mathematical thinking are considered and paid attention here.

There is no field of human activity in which computers are not used. They are used in factories, banks, railway reservation offices, national laboratories, military establishments and satellite launching stations etc. By the enormous use of computers we feel that we cannot bypass the computers any more. Whether we want them or not, we welcome them or not, computers will be there as they have come to stay. We must get acquainted with them and find out how to use them for our advantage.

When a computer is used to solve a problem, we have to use a particular programming language like FORTRAN, PASCAL, BASIC etc. to write a program. A program is a collection of instructions arranged in a proper order meant to solve a problem. The particular way of writing this set of logical instructions is called programming language. Without the knowledge of a programming language, we cannot use computers to solve a mathematical problem. Since our aim is to make students think logically and come up with a systematicstep-by-step procedure which, when followed, will lead to a solution (if it exists) of the problem. However, we will not be dealing with programming languages here. Discussion of these languages is outside the scope of this book. In this Chapter we develop algorithms and flowcharts.

In every topic in mathematics, we come across algorithms. For example, in the chapter on equations, we learn an algorithm of solving a quadratic equation $ax^2+bx+c=0$. In set theory, we have algorithms to find union, intersection, difference and complement of given sets. In commercial mathematics, we have algorithms to find simple and compound interests, profits and losses given % gain or loss and cost prices. In permutations and combinations, we have algorithms to find n_{c_r} and n_{p_r} given n and r. Many such examples can be cited where the concept of algorithms is used. The teachers can themselves give more problems to the students and encourage them to write algorithms and flowcharts.

It is clear that this topic is the heart of mathematics and therefore, it is necessary to study and teach it sincerely and seriously.

2. Content Analysis:

18.20 Algorithms

Given a problem, an algorithm is a set of steps arranged in a particular order, and when it is executed in that very order, it leads to the solution of the problem, if it exists. What is important here is the fact that an algorithm necessarily presupposes that a problem exists and that, in general, the problem has its solution. However, if the solution does not exists, then a conclusion to that effect is drawn and accordingly a statement is made. Another important aspect of an algorithm is that its steps are arranged in a particular order for their execution. The determination of order of steps requires logical thinking. It is this aspect which makes the teaching of this topic desirable and meaningful.

We can say that the modern meaning of algorithm is quite similar to that of recipe, process, method, technique, procedure routine, except that the word "algorithm" cannotes something just a little different. Besides merely being a finite set of rules which gives a sequence of operations for solving a specific type of problem, an algorithm have the following characteristics:

1. A given problem may not necessarily have a unique algorithm for its solution. In fact, a problem generally has more than one algorithm for finding its solution.

2. An algorithm should be precise.

3. It should be unambiguous.

4. Each step of an algorithm should be well defined.

5. Steps should be logically ordered.

6. It must end in a finite number of steps.

7. An algorithm has zero or more inputs i.e., quantities which are given to it initially before the algorithm begins execution.

8. An algorithm has one or more outputs i.e. quantities which have a specified relation to the input.

Now, we try to explain the above mentioned characteristics of an algorithm by taking few examples.

Example 1:

Given a set A containing first four natural numbers, Write an algorithm to find and enlist all the pairs of distinct elements of A such that the sum of their squares is even.

Solution: As mentioned above, a problem may not have a unique algorithm for its solution. Below we give two algorithms for finding the solution of this problem.

Algorithm 1:

Step 1: Take $i=1$

Step 2: Take $j=i+1$

Step 3: Let $A=i^2$

Step 4: Let $B=j^2$

Step 5: Let $S=A+B$

Step 6: If S is even,
then enlist the pair (i,j)

Step 7: If $j \angle 4$,
then increase the value of j by 1 and go to step 4.

Step 8: If $i=4$,
then stop;
else increase the value of i by 1
and go to step 2.

Algorithm 2:

Step 1: Take i=1

Step 2: Take j=i+1

Step 3: Let $A=i^2$

Step 4: Let $B=j^2$

Step 5: If A is odd and B is odd,
then enlist the pair (i,j)

Step 6: If A is even and B is even; then enlist the pair (i,j)

Step 7: If $j \angle 4$, then increase the value of j by 1 and go to step 4

Step 8: If i=4, then stop; else increase the value of i by 1 and go to
step 2.

Example 2:

Write an algorithm to print first 100 positive integers.

Solution:

Algorithm 3:

Step 1: Take i=1

Step 2: Increase the value of i by 1

Step 3: If i=100, then point i and step; else go to step 2.

Let us check whether by this algorithm our aim is achieved or not. Now we carry out the working of the above three steps.

1. i =1
2. i becomes 2
3. As $i \neq 100$, in step 3, we go to step 2.
4. i becomes 3.
5. As $i \neq 100$ in step 3, we go to step 2.
6. i becomes 4.
7. We see that as soon as i = 100, the value of i(which is 100) will be printed and the execution of the algorithm

Has our aim been achieved ? Surely, not. Instead of printing all the first 100 positive integers, it has printed the 100th positive integer only. Clearly, our algorithm 3 is not precise as it does not give us the solution of the problem. By using the following alternative algorithm, we get the solution of the problem.

Algorithm 4:

Step 1: Take i = 1

Step 2: Print i

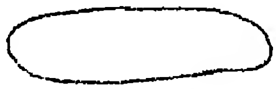
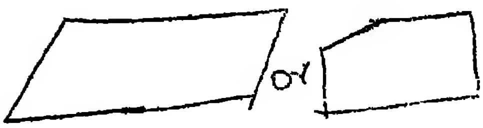
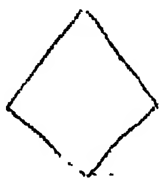
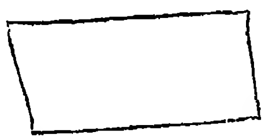
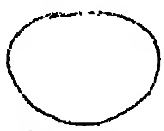
Step 3: Increase the value of i by 1

Step 4: If i = 101
then stop
else go to step 2.

It is clear that as soon as i becomes 101, the execution of the algorithm stops and before it stops, all the 100 integers are printed.

18.3 Flowcharts

A flowchart is a graphical representation of an algorithm, i.e. it is a visual picture of the logical steps and the flow of control between the various steps of an algorithm. As already explained, an algorithm is a step-by-step method for solving a problem. In a flowchart, we enclose each operation, instruction or series of instructions in a box and indicate the flow of control by arrow between the boxes. Furthermore, different types of operations are indicated by differently shaped boxes as indicated below:-

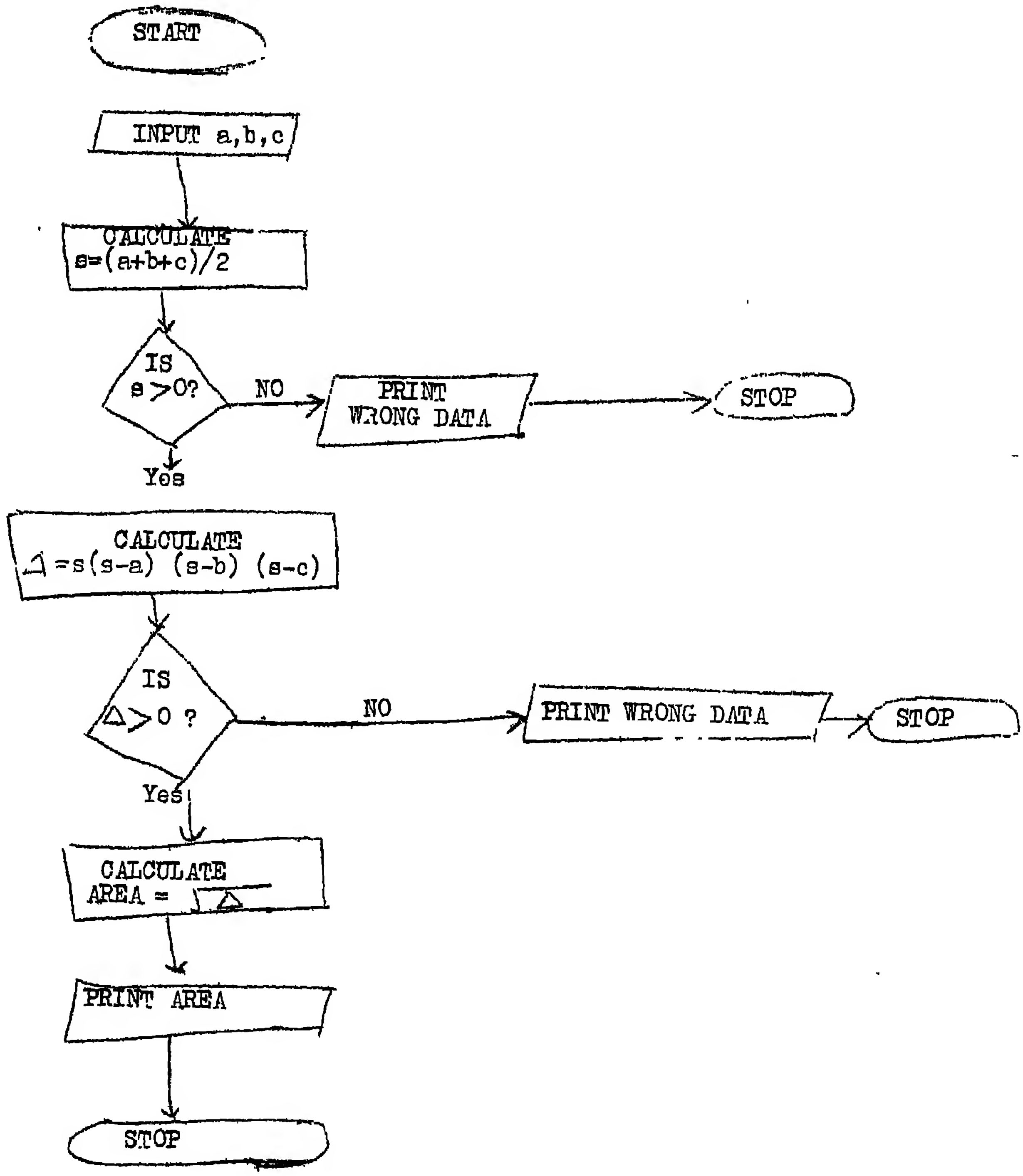
Shape of the box	Meaning
 A stretched ellipse	For start or stop
 A parallelogram	For input or output: The datafed into the computer and the print out given by the computer
 Diamond	For a decision. Computer has to decide for an alternative instruction.
 A rectangle	For a calculation or process other than a decision.
 A circle	For a connection between two parts of a flowchart.

The symbol " $\leftarrow$ " is called an assignment symbol. $N \leftarrow N+1$ means that N has been assigned the new value N+1. Sometimes the symbol "=====" is also used as an assignment symbol. We agree to use the symbol "$\leftarrow$" as an assignment symbol or for the replacement operation.

Now, we learn the study of flowcharts through examples.

Example 3: Draw a flowchart to find the area of a triangle when its three sides a, b, c are given:

Solution:

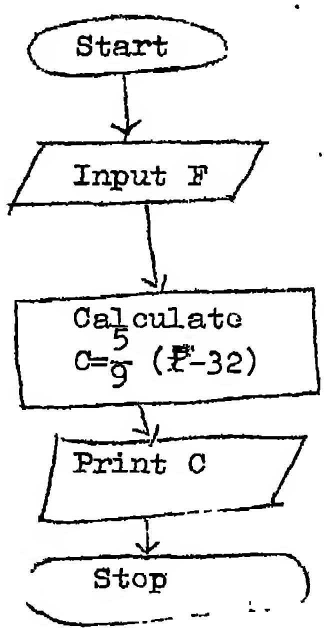


Example 4:

Draw a flowchart to convert a Fahrenhit temperature F to centigrade temperature C using the formula,

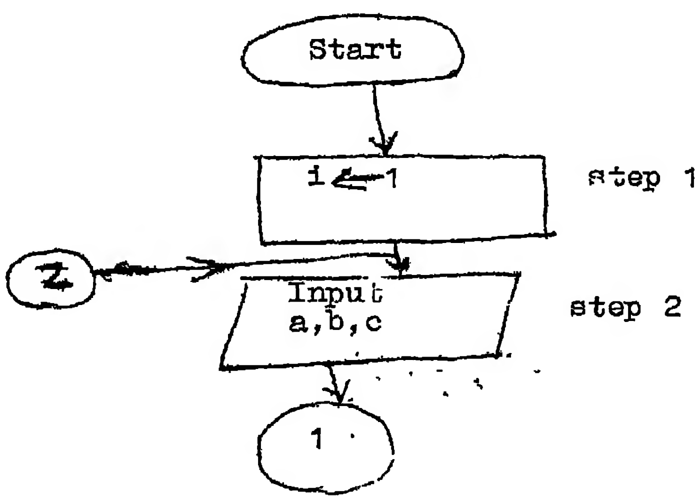
$$C = \frac{5}{9}(F-32)$$

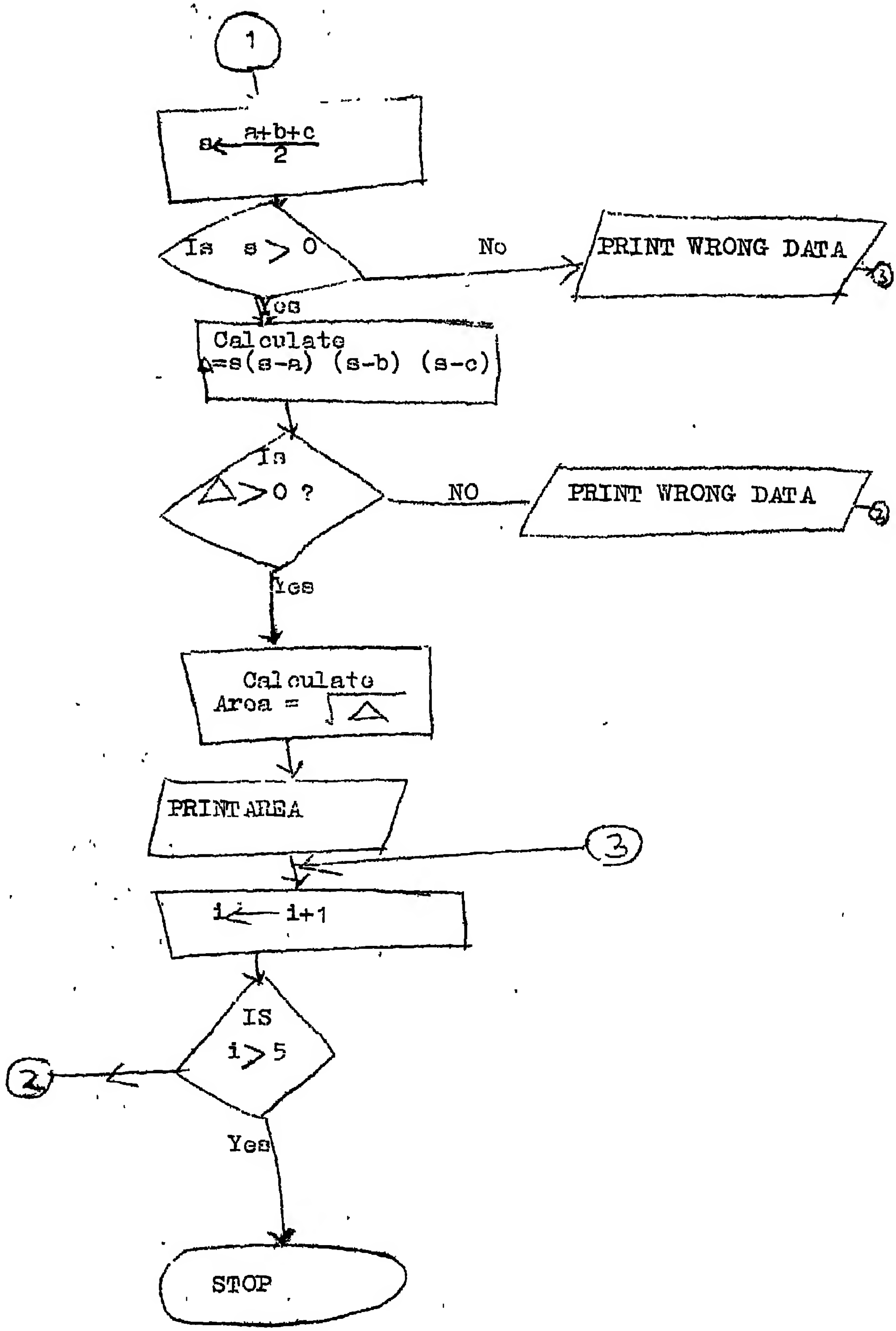
Solution



Example 5:

Three sides a,b,c of each of five triangles are given. Draw a flowchart to find the areas of five triangles.




1
s ← (a+b+c)/2
Is s > 0
No
PRINT WRONG DATA
3
Yes
Calculate
Δ=s(s-a) (s-b) (s-c)
Is
Δ>0 ?
NO
PRINT WRONG DATA
3
Yes
Calculate
Area = √Δ
PRINT AREA
3
i ← i+1
IS
i> 5
2
Yes
STOP

18.4 Loops

The flowchart of Example 5 in the formof an algorithm is given below:

Algorithm 5

Step 1 : Take $i = 1$

Step 2 : Input $a, b, c,$

Step 3 : Calculate $s = \frac{a+b+c}{2}$

Step 4 : Calculate Area = $\sqrt{s(s-a)(s-b)(s-c)}$

Step 5 : Print area

Step 6 : Increase the value of i by 1

Step 7 : If $i > 5$,
then stop;
else go to step 2.

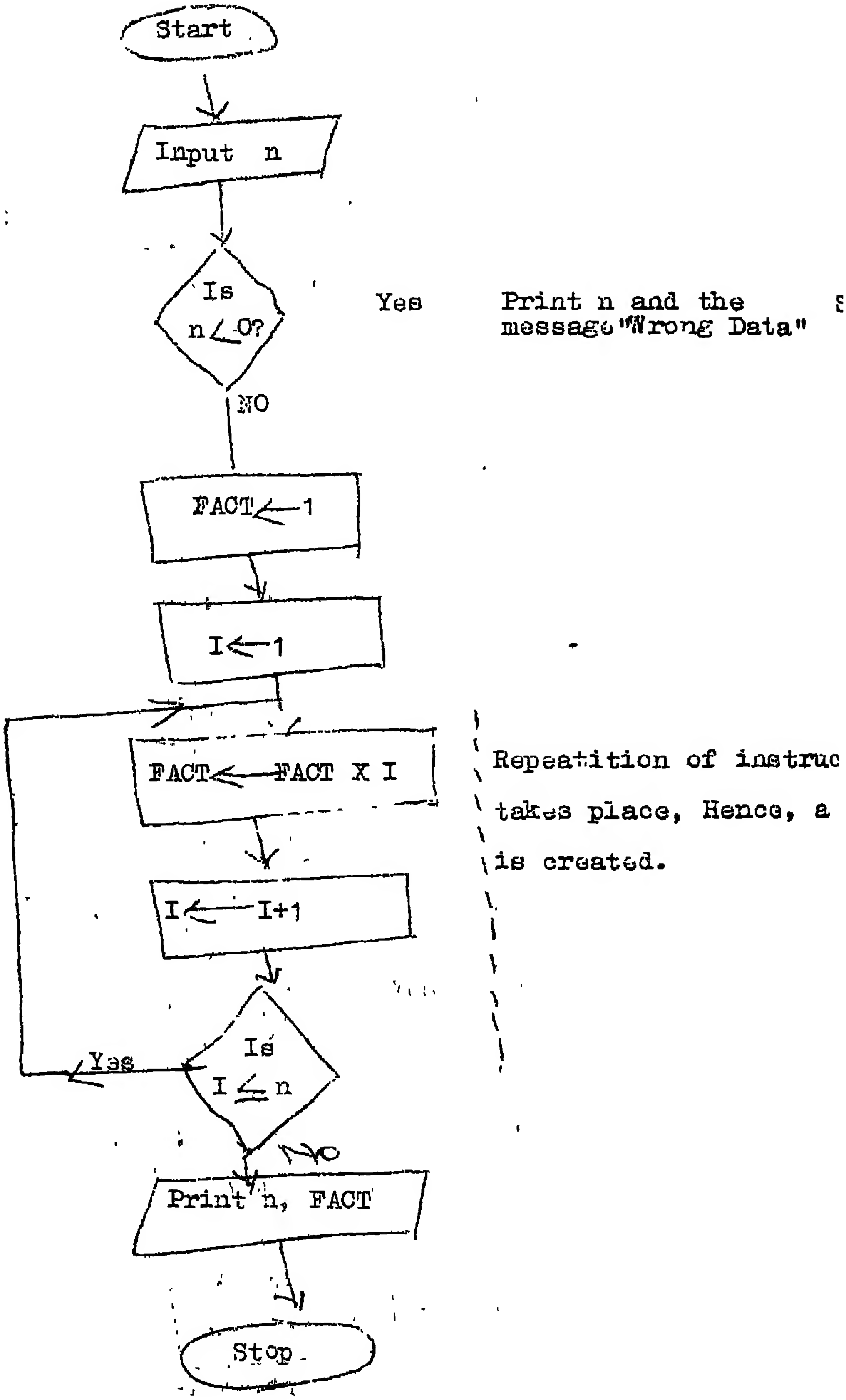
Observe that the steps from step 2 to step 7 are repeated many times. When a set of instructions (steps) is repeated many times, it is called a loop. Hence, in the algorithm 5, the set of steps – step 2 to step 7 – has created a loop. Every time the loop is repeated, the value of i is increased by 1. As soon as the value of i becomes greater than 5 (i.e. becomes equal to 6) the exit from the loop takes place to stop execution of the algorithm.

To understand more about loop, we work out few more examples.

Example 6

Draw a flowchart to find the value of $n!$ for a given positive integer. n.

Solution


Start
Input n
Is
n < 0?
Yes
Print n and the
message "Wrong Data"
NO
FACT ← 1
I ← 1
FACT ← FACT X I
Repeatition of instruc
takes place, Hence, a
is created.
I ← I+1
Is
I ≤ n
Yes
NO
Print n, FACT
Stop

Example 7

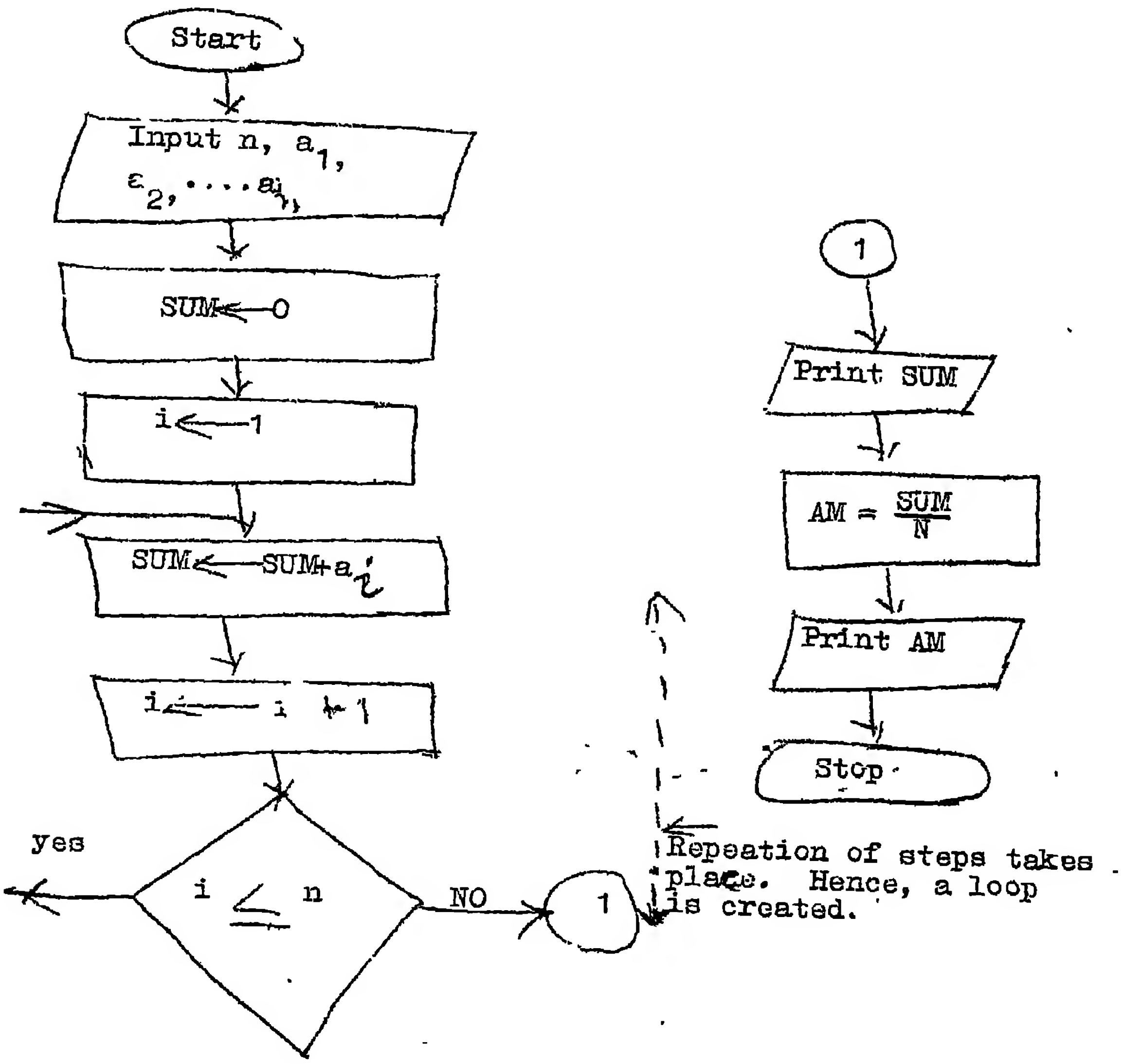
Draw a flowchart to find the arithmetic mean of the given numbers.

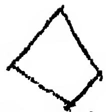
Solution:

Let the given numbers be a_1, a_2, a_3......, a_n where n is any given positive integer. The arithmetic mean, denoted by AM, is given by

$$AM = \frac{SUM}{n}$$

where $SUM = a_1+a_2+.......+a_n$. The following is the flowchart:



Note: The box  being a decision box, instead of writing 'is $i \leq n$?' as mentioned in the textbook, we agree to write simply '$i \leq n$' in the above mentioned box in the above flowchart,

3. LEARNING OUTCOMES

(a) Essential learning outcomes for all

i) Students should acquire ability to identify such problems from their fields of study as are amenable to step-by-step execution. In general, students' ability of writing a solution procedure logically, using various steps, will be sharpened.

ii) Students should be able to analyze a problem and split it up into sub-problems such that each subproblem is simpler than the original problem.

iii) Students should be able to identify the set of actions and the order of therein execution for obtaining a solution.

iv) Capacity of logical thinking should be developed among students.

b) Learning outcomes for the higher groups

i) Students should be able to tackle more complex problems developing algorithms to solve them.

ii) Students should develop the ability of redefining the problem transforming it into a form which may be more convenient for obtaining a solution.

iii) Students should be able to recognize the most precise, simple and reliable algorithm out of several algorithms of solving a given problem.

iv) Inmate capacity for logical, thinking present in a group of students should be enhanced.

4. Teaching Strategies

a) Motivation for the development of concepts

We emphasize and explain below some ideas and terms that are sometimes not properly understood and assimilated.. The discussion here will help teachers to appreicate the significance of these usages with a view to communicate them to the students.

18.1 Computers

First of all, the meaning of the word 'computer' itself should be clearly understood. What is a computer ? One is likely to say that "any device that aids to perform computations is a computer" Now consider the following device to solve the equation

$$x^3 + x = 20 \quad \text{--- --- --- --- --- ---} \quad (1)$$

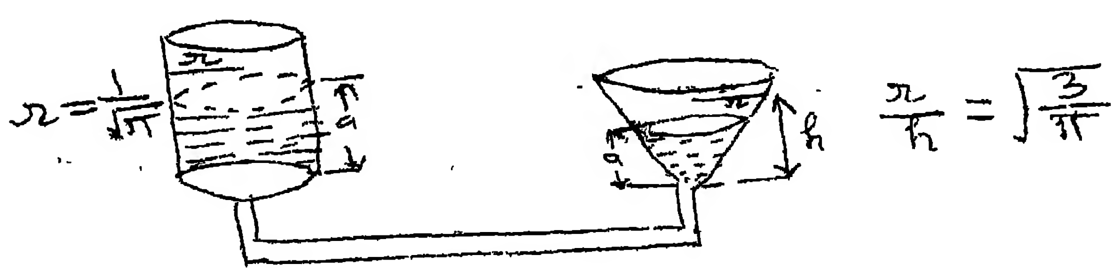


Fig. 18.1

Device to solve $x^3+x=k$, k is a positive number consider, on the left, a circular cylinder (made of aluminium or plywood or cardboard sheet etc.) with radius of its circular cross-section = $\frac{1}{\pi}$. Similarly, take a right circular cone on the right such that the ratio $\frac{r}{h}$ (ratio of radius of a circular cross-section to its height from the vertex of the cone)= $\sqrt{\frac{3}{\pi}}$ at any place of the cone. The lower portion of the cylinder and cone are connected by a tube as shown in the Fig. , and this tube is always kept filled with water. Now, take 20 C.C. water and pour it either in the cylinder or cone from the top. Because of the property of water, it will stand at the same height, say a, in both cylinder and cone Now,

Volume of water in cylinder= $\pi r^2 a = \pi(\frac{1}{\pi})a = a$ c.c.

Volume of water in cone = $\frac{1}{3}\pi r^2 a = \frac{1}{3}\pi(\frac{3}{\pi}h^2)a$

$= a^2.a = a^3$ c.c.

So volume of the water poured = 20 c.c. = (a^3+a) c.c.

Thus a satisfies equation (1) that was to be solved. Therefore, a is a solution of the equation (1). Thus the device given in Fig. 18.1 helps in computing the solution of the equation of the type $x^3+x=k$, where k is a positive real number.

The device given in Fig. 18.1 is an analog computer. The main characteristics of analog computers are:

1. Numbers are represented by a continuously varying quantity like distance, voltage, current etc. For example, in the device of Fig.18.1, numbers are represented by a distance giving the height of water column in the cyliner or cone.

2. The accuracy of computation performed is limited. This is because of the difficulty in the measurement of a continuous quantity to represent a specific number.

3. They are meant to solve a particular class of problems. For example, the device of Fig. 18.1 solves any equation of the type $x^3+x=k$, $k \in \mathbb{R}^+$ only. It cannot be used to solve any other equation even simpler than this, e.g., one that is quadratic.

When we use the term 'computer' for a device, what is understood is that the device must have the following characteristics:

1) It is electronic (e.g. T.V. is an electronic device).

2) It is digital (e.g. you have seen the digital watches).

3) stored-program-concept is employed.

So remember that a computor is a machine that is electronic, digital and in which stored-program-concept is used. Stored-program-concept means that along with data/information of the problem, the set of instructions that forms a program are also stored in the main memory (explained below) of the computer. This idea looks very simple now-a-days but it was a revolutionary idea which changed the complexion of mathematical computation to a great extent. Credit of this idea goes to John Von Neuman (1903-1957) and Alan Turning (1912-1954).

From the above discussions, we can conclude that the device given in Fig. 18.1 is not a computer.

18.2 Algorithms

With the extensive use of computers, one may tend to feel that algorithms are always associated with computers. But this is not true. The word algorithm was most frequently associated with "Euclid's algorithm", a process for finding the greatest common divisor of two numbers. Now, we exhibit Euclid's algorithm:

EUCLID'S ALGORITHM

Given two positive integers m and n, find their greatest common divisor, i.e., the largest positive integer which divides both m and n.

Step 1:(Find remainder) Divide m by n and let r be the remainder. (we will have $0 \leq r < n$).

Step 2:(Is it zero?) If r=0, the algorithm terminates; n is the answer.

Step 3:(Interchange): Set $m \leftarrow n$, $n \leftarrow r$ and go to step 1.

Let us work out Euclid's algorithm taking m=25 and n=15.

Step 1:Dividing m by n, we get the remainder r=10.

Step 2: Since $r \neq 0$, therefore, we go to step 3 of the algorithm.

Step 3: Here m=15 and n=10 and we go to step 1 of the algorithm.

Step 4: Dividing m by n, we get the remainder=5.

Step 5: $r \neq 0$; therefore, we go to step 3 of the algorithm.

Step 6: Here $m=10$ and $n=5$, and we go to step 1 of the algorithm.

Step 7: Dividing m by n, we get the remainder $r = 0$.

Step 8: $r=0$; therefore, the algorithm terminates and $n=5$ is the answer.

b) MYSCONCEPTIONS/COMMON ERRORS

i) Many books mention that ENIAC (Electronic Numerical Integrator and Computer) which was built in 1946 is the first computer. This is not correct because stored - program-concept was not used in it. EDSAC (Electronic Delayed Storage Automatic Computer) in the first computer which was built in 1949.

ii) Computers main memory can be looked upon as a collection of compartments or locations, from the users' point of view. Each compartment is generally called a word. Each word has a number attached to it called its address as shown in Fig. 18.2. These addresses start from 0 and go through 1, 2.......

0	1	2
3	4	5
6	7	8
9	10	11
12	13	14

Computer's main memory from user's point of view

Fig. 18.2

iii) In ordinary usage, we use the world computation when numbers are involved. On the numbers, we perform the operations of addition, multiplication etc. This is called numeric computation. Suppose we are given 1000 names; they are to be arranged in the alphabetical order. The work involved in achieving this is called non-numeric computation. In computing field, the term computation refers to both numerica and non-numeric computation.

iv) While presenting algorithms as well as flowcharts, the symbol '$\leftarrow$' is used as an assignment symbol. "$i \leftarrow 1$" is read as "i becomes 1" and "$j \leftarrow j+1$" is read as "j becomes j+1". Note that the arrow always appears pointing to the left. Further to the left of $\leftarrow$, only one variable name can appear and on the right of $\leftarrow$ any expression involving variables and operations can appear. Thus "$a \leftarrow n^2-n+1$" is a valids statement. Such a statement is called an assignment statement. On the otherhand, "$a+1 \leftarrow b^2-b+2$" is not a valid statement as an expression (viz. a+1) appears on the left of $\leftarrow$.

An assignment statement is executed as follows. The expression on the right of $\leftarrow$ is evaluated with current values of the variables appearing therein and the value of expression thus calculated is assigned as the new value to the variable appearing on the left of $\leftarrow$ and whatever value, if any, this variable on the left of $\leftarrow$ had earlier to the execution of this assignment statement, is wiped out and is not available as the value of that variable hereafter in the algorithm.

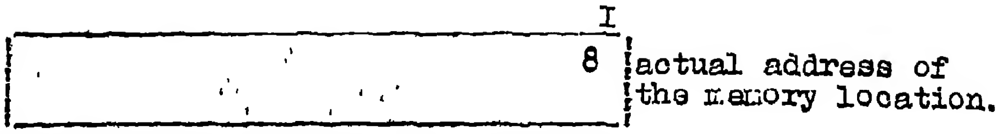
Consider the two consecutive assignment statements:

I $\longleftarrow$ 1..............................(2)

I $\longleftarrow$ I+1...........................(3)

Statement (2) assigns the value 1 to the variable I. When statement (3) is executed, we first evaluate the expression I+1 appearing on the right of $\longleftarrow$. At the time of evaluation of I+1, the value of I is 1 since statement (2) has done that assignment, and it is not changed before the execution of (3) is undertaken. Therefore, when (3) is executed, I+1 appearing on the right of $\longleftarrow$ assumes the value 1+1=2. Now this value 2 is assigned to I as it appears on the left of $\longleftarrow$. Thus new value of I becomes 2 and its old value (namely 1) is completely lost. For the portion of the algorithm that appears after these two statements, value of I will be 2 till it is changed by some other statement.

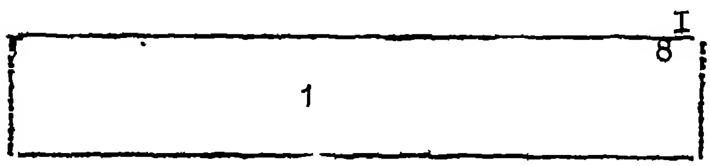
Actually, what happens inside a computer when assignment statements are executed is interesting to note and is explained below. When a computer executes a program (i.e. an algorithm written in a particular programming language), it reserves one separate memory location for each of the variables occuring in the program. As soon as a variable is assigned a value, this value is put in the memory location reserved for it. Whenever the value of this variable is changed in further part of the algorithm by another assignment statement, the new value of the variable is put in the memory location reserved for the variable and the old value present in that location is automatically lost. For example, consider the statements (2) and (3) given above. Consider the memory location reserved for the variable I. Suppose it looks as shown in Fig. 18.3



Memory location reserved for variable I

Fig. 18.3

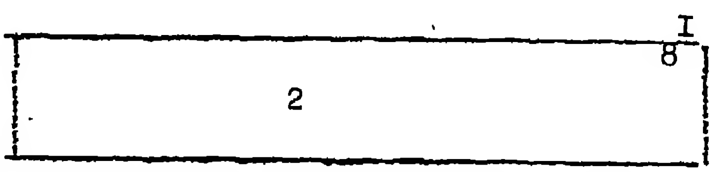
When (2) is executed, value 1 is put in the memory location, so memory location will look as shown in Fig. 18.4



Memory location reserved for I after(2) is executed

Fig. 18.4

After (3) is executed, the value of I becomes 2. So after(3) is executed, the memory location reserved for I looks as shown in Fig.18.5.



Memory location for I after (3) is executed

Fig. 18.5

Note that after the execution of(3), earlier contents of the memory location for I are lost.

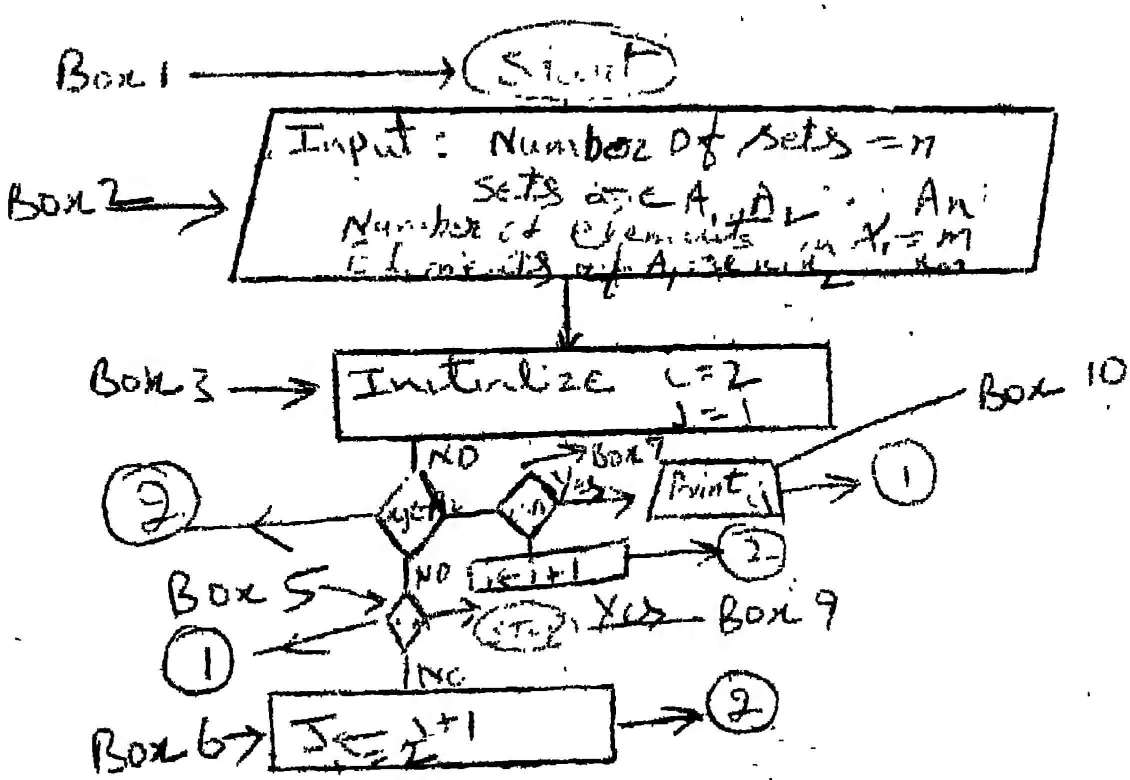
Note that after the execution of (3), earlier contents of the memory location for I are lost.

V. **Flowchart** is a pictorial representation of an algorithm, Generally it is said that a picture conveys the information many times better than the words. Flowchart exhibits the structure of an alogrithm.

It is an experience that programmes (persons who are experts in writing an alogirthm in the form of a programming language) write the algorithm or program first and then produce flowchart, if demanded. Here teachers are advised to ecnourage the students to write flowcharts first; this will help them to find the precise logical steps to be followed for getting the solution of a problem.

C. Explanation of examples from the text-book.

1. Example 18.4 the textbook given a flowchart to find interesection of a finite number of finite sets. The flowchart given there is



We have numbered the boxes. We carryout the instructions of this flowchart by taking the example given in the textbook.

The given sets are $A=\{1,2,3,4,5\}$,

$B=\{3,4,5,8,9\}$,

$C=\{2,3,8,9\}$

$D=\{1,2,3,10\}$

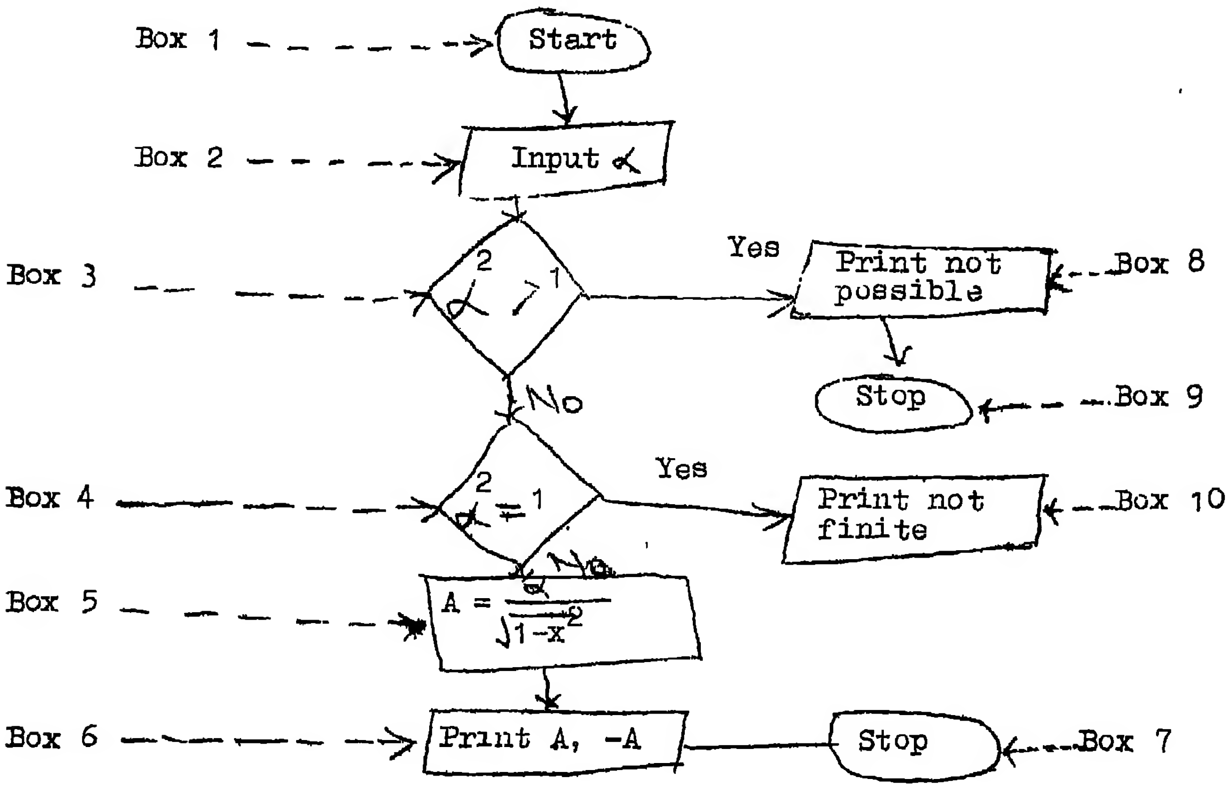
We want to find out $A\cap B\cap C\cap D$.

First, we rename the sets as $A_1=\{2,3,8,9\}$, $A_2=\{1,2,3,4,5\}$, $A_3=\{3,4,5,8,9\}$ $A_4=\{1,2,3,10\}$ so that the set A_1 has least number of sets. Now, we present our working procedure associated with the execution of the flowchart systematically in tabular form.

S.No.	Box No. visited	type of box	Action performed
1.	1	start	execution of algorithm starts
2.	2	Input	$A_1=\{2,3,8,9\}$, $A_2=\{1,2,3,4,5\}$, $A_3=\{3,4,5,8,9\}$, $A_4=\{1,2,3,10\}$, $n=4$, $m=4$, $x_1=2$, $x_2=3$, $x_3=8$, $x_4=9$.
3.	3	Computation	$i=2$, $j=1$
4	4	decision	yes, because $x_1 \in A_2$
5	7	decision	No, because $i=2\neq 4=n$
6.	8	computation	$i=2+1=3$. Here the value of i becomes 3.
7.	4	decision	No, because $x_1 \notin A_3$
8	5	decision	No, becuase $j=2 \neq 4=m$
9.	6	Computation	$j=1+1=2$, $i=2$
10.	4	decision	Yes, x_2 $(=3) \in A_2$
11.	7	decision	No, because $i=2\neq 4=n$
12.	8	computation	$i=2+1=3$
13.	4	decision	Yes, because $x_2(=3) \in A_3$
14.	7	decision	No, because $i=3\neq 4=n$
15.	8	computation	$i=3+1=4$
16.	4	decision	Yes, because x_2 $(=3) \in A_4$
17.	7	decision	Yes, because $i=4=4=n$
18.	10	output	$\boxed{\text{3 is printed}}$
19.	5	decision	No, because $j=2\neq 4=m$
20.	6	computation	$j=2+1=3$ and $i=2$
21.	4	decision	No, because $x_3(=3) \notin A_2$
22.	5	decision	No, because $j=3\neq 4=m$
23.	6	computation	$j=3+1=4$, $i=2$
24.	4	decision	No, because $x_4(=9) \notin A_2$
25.	5	decision	yes, because $j=4=4=m$

After the execution stops we observe that our answer to the problem is 3 which was printed at S.No. 18. Let us see the roll of iand j. $x_j \in A_i$ represents the belonging of j^{th} element of the set A_1 to the other sets (i.e. A_2, A_3, A_4). Therefore, variation of j keeps the track of the elements of the set A_1 and variation of i keeps the track of different sets (in our case they are A_2, A_3, A_4). Observe that the value of j will vary from 1 to 4 and that of i will vary from 2 to 4.

2 Example 18.7 in the textbook gives a flowchart which is not correct. The correct flowchart is as follows:



Let us workout the steps of this flowchart to find the solution to 3(a) of Exercise 18.1 in the textbook.

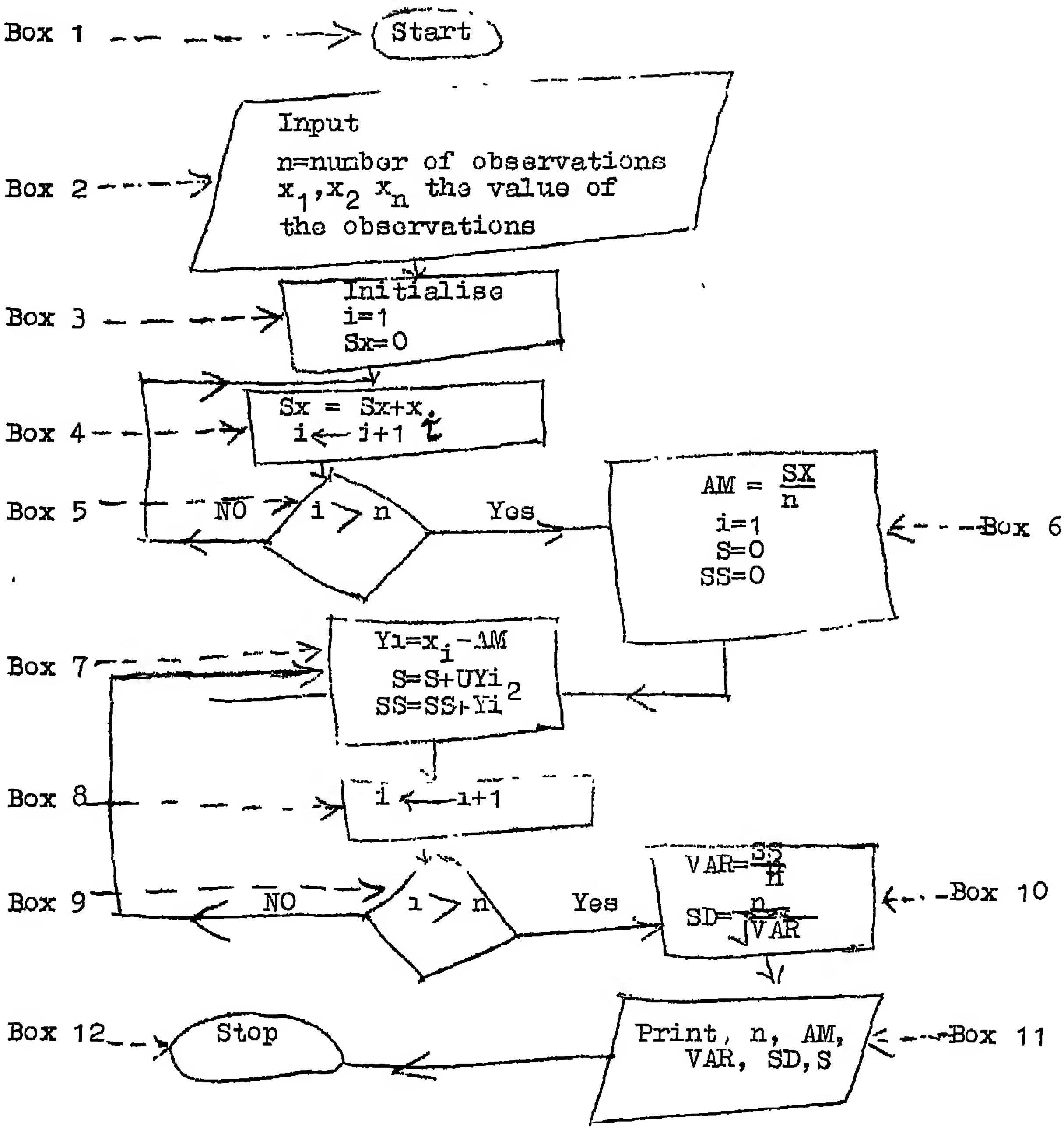
Since $\sin\theta = \frac{1}{2} \implies = \frac{1}{2}$

S.No.	Box No. visited	Type fof Box	Action performed
1.	1	start	execution of the algorithm starts
2.	2	Input	$\alpha=\frac{1}{2}$
3.	3	decision	No, because $\alpha^2=\frac{1}{4} \not> 1$
4.	4	decision	No, because $\alpha^2=\frac{1}{4}\neq 1$
5.	5	computation	$A=\frac{1}{\sqrt{3}}$
6.	6	Output	$A=\frac{1}{\sqrt{3}}$ and $A=-\frac{1}{\sqrt{3}}$

$\therefore$ $\tan\theta=\frac{1}{\sqrt{3}}$ and $\tan\theta=-\frac{1}{\sqrt{3}}$,

is the solution to the given problem.

3. Example 18.9 in the text book gives a flow chart to find the variance and standard deviation for the given data. This flowchart is wrong. The correct flowchart is given below:



The following points will make clear the working procedure in the flowchart:

a) Steps of the flowchart from Box 1 to Box 6 will give us the arithmetic mean value of the given data as explained in Example 7. AM denotes the arithmetic mean and SX denotes the sum of the values of the given data.

c) Steps of the flowchart from Box 7 to Box 10 will give us

$$S=(x_1-AM)+(x_2-AM)+(x_3-AM)+\ldots\ldots\ldots+(x_n-AM)$$
$$SS=(x_1-AM)^2+)x_2-AM)^2+(x_3-AM)^2+\ldots\ldots\ldots+(x_n-AM)^2$$
$$VAR=\frac{SS}{n}$$
$$SD=\sqrt{VAR}$$

d) As soon as the execution of the alogrithm is stopped, values for n, AM, VAR, SD, S are printed for our purpose.

Note that the value of i=1 at Box 6 was not changed in the flowchart given in the textbook . Therefore, the value available for i at the decision box following Box 7 will always be 1 and the execution of the flowchart will never stop. To overcome this difficulty the box $i \leftarrow i+1$ following the Box 7 has been added. Also note that the value of S has not been used to find out variance and standard deviation. The teachers can drop the statements S=0, S=S+Yi from the Boxes 6 and 7 respectively. Similarly, the S can be droped from the Box 11.

d. ANSWERS AND SOLUTIONS TO SELECTED PROBLEMS IN EXERCISE 18.1 OF THE TEXTBOOK.

1. (a) (i) 3, (ii) 2, (iii) 1,
   (b) 4

2. (a) Stage 1:

$$\left.\begin{array}{l} n \leftarrow 10 \\ r \leftarrow 6 \end{array}\right\} \text{Input}$$
$$r \leftarrow 4 \quad \text{as } r > \frac{n}{2} \Rightarrow r \leftarrow 10-6$$
$$A \leftarrow 10$$
$$B \leftarrow 1$$
$$C \leftarrow 10$$

Stage 2: $A \leftarrow 9$

$B \leftarrow 2$

$C \leftarrow 45$

$B=1 \neq r=4$

Stage 3: $A \leftarrow 8$

$B \leftarrow 3$

$C \leftarrow 120$

$B=3 \neq r=4$

Stage 4: $A \leftarrow 7$

$B \leftarrow 4$

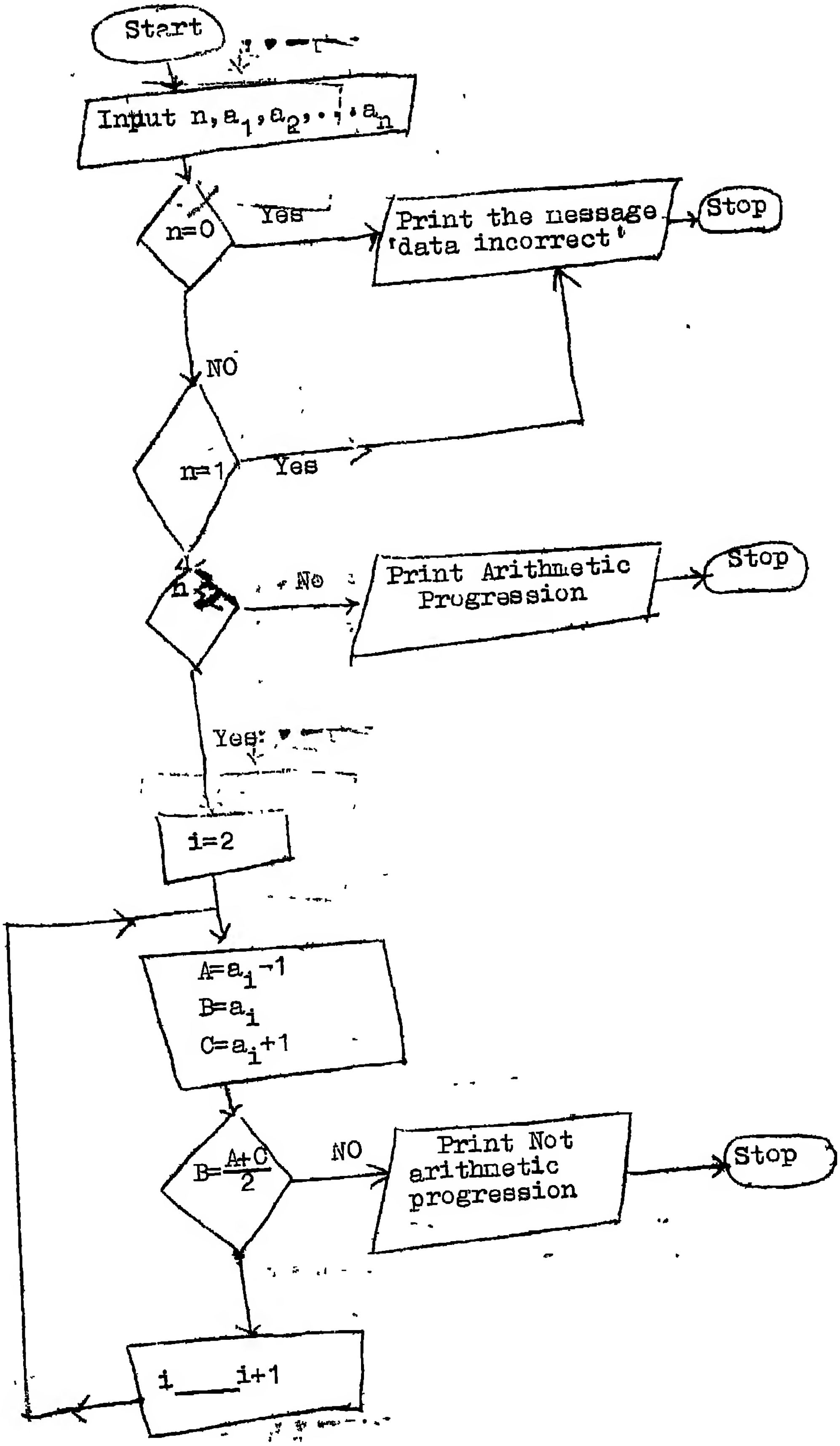
$C \leftarrow 210$

$B=4 = r=4$

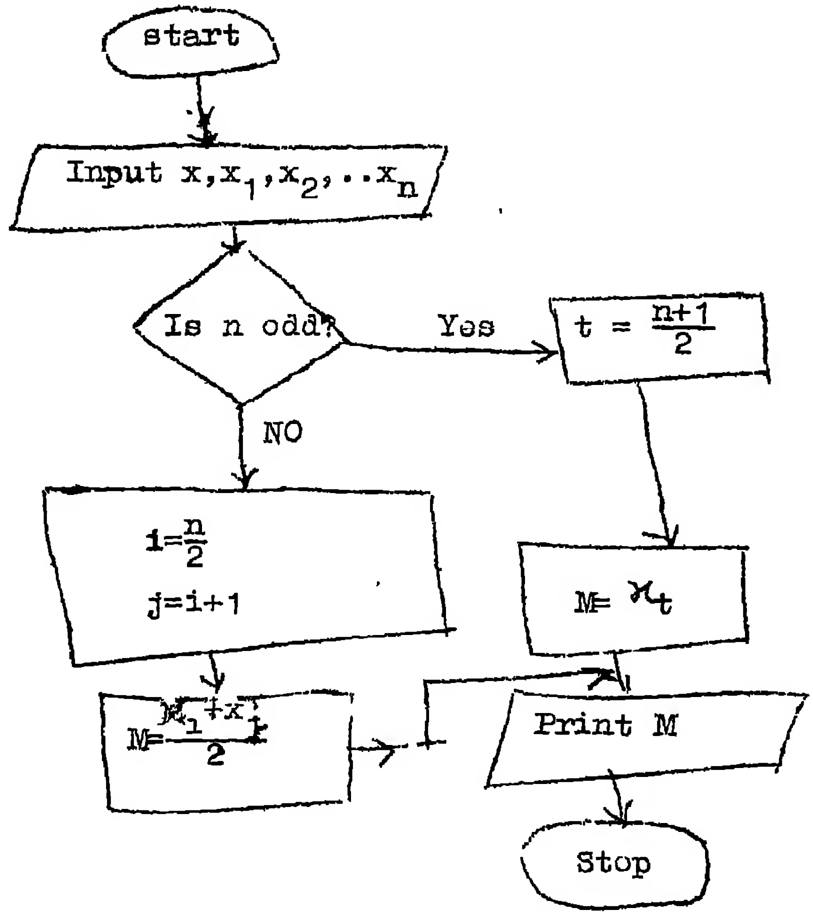
$\therefore \quad C = 210$

3 (a) $\frac{1}{\sqrt{3}}, \quad \frac{-1}{\sqrt{3}}$

(c)


Start
Input n, $a_1, a_2, \ldots, a_n$
n=0
Yes
Print the message 'data incorrect'
Stop
NO
n=1
Yes
n=2
No
Print Arithmetic Progression
Stop
Yes
i=2
$A=a_{i-1}$
$B=a_i$
$C=a_{i+1}$
$B=\frac{A+C}{2}$
NO
Print Not arithmetic progression
Stop
i ___ i+1

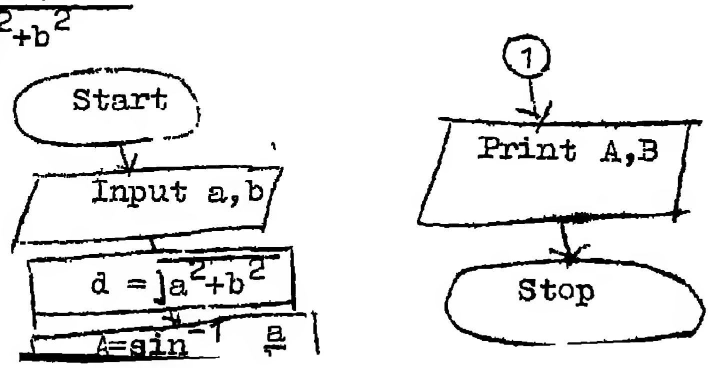
(d) First arrange the data in the ascending or descending order. We denote this arranged data by $x_1, x_2, \ldots\ldots x_n$. M denotes the median of the data.



e) We know that $\frac{\text{Sin } A}{a} = \frac{\text{Sin } B}{b} = \frac{\text{Sin } C}{c}$ Let $\angle C = 90^0$

$\therefore \angle A = \sin^{-1} \frac{a}{\sqrt{a^2+b^2}}$

$\angle B = \sin^{-1} \frac{b}{\sqrt{a^2+b^2}}$



5. CHAPTER TESTS

1. Here is a collection of some more exercises.

1. Identify some situations, from your daily life, for whihh algorithms can be written down.

2. From each chapter of your mathematics book, identify the problems, the solution methods of which can be presented in the form of algorithms.

3. Write an algorithm which will print each two digit odd number N. its square N^2 and its cube N^3.

4. Write an algorithm which will print the sum of odd numbers, each number having two digits.

5. Write an algorithm to find the sum of the numbers $a_1, a_2 \ldots a_n$ and the sum of the squares of these numbers.

6. Write an algorithm and its corresponding flowchart

   a) To print even positive integers upto 100

   b) To count the number of positive integers and negative integers when a sequence of integers $a_1, a_2 \ldots a_n$, where a_i is positive or negative, is given.

7. A sequence of numbers $a_1, a_2 \ldots\ldots a_n$ is given. Write an algorithm and its corresponding flowchart to find another sequence $b_1, b_2 \ldots\ldots b_n$ such that $b_1 = a_n$, $b_2 = a_{n-1}$, $b_n = a_1$.

8, Write an algorithm to print the Fibonacci sequence 1,1,2,3,5,8,13,21,.......having 50 terms. Also draw its flowchart.

9. Draw a flowchart to find and print the value of B given by

$$B(x) = \begin{cases} x^2+9, & x < 0 \\ x^4+3x^2+5, & 0 \leq x < 20 \\ 12x+7 & x \geq 20 \end{cases}$$

Appendix - I

INTRODUCTION TO VEDIC MATHEMATICS

PART-I

1. INTRODUCTION

Ancient Vedic Mathematics system has been re-expounded in recent times by Sankaracharya Swami Sri Bharati Krishna Tirtha Ji. Vedic Maths (VM) offers a new approach to mathematics based on pattern recognition and allows for constant expression of a student's creativity. VM provides a multiple choice system with flexibility at each stage of working, which keeps the lively and alert. VM provides mental and superfast methods alongwith quick grosschecking systems. As such VM is a boon for all competitions. VM has been found to be easier to learn and delightful to use.

operations on matrices and determinants and problems of statistics etc involve large number of arithmetical operations. VM allows for combined operations of arithmetic, which can lead to large savings in time and space of working. Further, we can do every operation from left to right and thus obtain the most significant digit/digits first and directly.

Let us briefly learn the VM methods for basic operations. Further details are available elsewhere (Ref. 1,2,3,4).

contd..

2. URDHVA SUTRA

The Urdhva Sutra is the third in the total list of sixteen sutras. It states 'Udhva Tiryagbhyam', which means Vertically and Crosswise. This sutra provides the general methods of multiplication and has large number of applications (Ref.2, 4 & 5).

Example 1: For the produce of two 1-digit numbers, we simply multiply the digits vertically.

$$\begin{array}{r} 6 \\ \times\ 4 \\ \hline 24 \\ \hline \end{array}$$

Example 2: For multiplying two numbers, each with 2 digits:

We 1. Take the vertical produce of the right column digits (units)

$$\begin{array}{rl} & 1 \quad 2 \\ \times & \underline{3 \quad 1} \\ & 3/7/2 = 372 \end{array}$$

2. Take the cross product: $2x3+1x1 = 7$

3. Take the vertical product of the left column digits (tens) : $1 \times 3 = 3$

Example 3: Similarly form the product of

$$\begin{array}{rl} & 2 \quad 3 \\ \times & \underline{4 \quad 2} \\ \hline \end{array}$$

contd..

The products are

1. $\begin{matrix} 2 & \left[\begin{matrix} 3 \\ 1 \end{matrix}\right] \\ 4 & \end{matrix} = 3 \times 1 = 3$ i.e.

2. $\left[\begin{matrix} 2 & 3 \\ 4 & 1 \end{matrix}\right] = 2 \times 1 + 3 \times 4 = 14$

3. $\begin{matrix} \left[\begin{matrix} 2 \\ 4 \end{matrix}\right] & \begin{matrix} 2 \\ 1 \end{matrix} \end{matrix} = 2 \times 4 = 8$ i.e

$$\begin{array}{ccc} 2 & & 3 \\ \underline{4} & & \underline{1} \\ 8 / & 4 / & 3 \\ \underline{1} & & \\ 9 & 4 & 3 \end{array} = 943$$

Example 4: Let us compute the produce 132 x 4Ø5

We have five steps of working

1. $\begin{matrix} 1 & 3 & \left[\begin{matrix} 2 \\ 5 \end{matrix}\right] \\ 4 & Ø & \end{matrix} = 2 \times 5 = 1Ø$

2. $\begin{matrix} 1 & \left[\begin{matrix} 3 & 2 \\ Ø & 5 \end{matrix}\right] \\ 4 & \end{matrix} = 3 \times 5 + 0 \times 2 = 15$

3. $\left[\begin{matrix} 1 & 3 & 2 \\ 4 & Ø & 5 \end{matrix}\right] = 1 \times 5 + 2 \times 4 + 3 \times Ø = 13$

4. $\begin{matrix} \left[\begin{matrix} 1 & 3 \\ 4 & Ø \end{matrix}\right] & \begin{matrix} 2 \\ 5 \end{matrix} \end{matrix} = 1 \times Ø + 3 \times 4 = 12$

5. $\begin{matrix} \left[\begin{matrix} 1 \\ 4 \end{matrix}\right] & \begin{matrix} 3 & 2 \\ Ø & 5 \end{matrix} \end{matrix} = 1 \times 4 = 4$

contd..

The extra digit is carried forward.

	1	3	2
x	4	Ø	5

4/2/3/5/Ø

1/1/1/1/

5 3 4 6 Ø = 5346Ø

Discussions:

1. Simplifications of 2 digits in a single column is the vertical product (steps 1 and 5).

2. Simplification of 4 digits in 2 columns is the crosswise product (steps 2 and 4).

3. Simplifications of 6 digits in 3 columns is the crosswise product of corner digits and vertical product of central column digits.

4. The steps can be easily reversed, and thus the result shall be obtained from left to right.

5. The carry digit can be directly added.

The rationale of the Vedic Method can be easily seen if we consider the multiplicationof the following 2 polynomials.

Example 5: $1x^2+3X$ $+2$ $1ØØ+3Ø+2 = 132$

$4x^2+Øx$ $+5$ $4ØØ+ØØ+5 = 4Ø5$

$4x^4/+12x^3/+13x^2/+15x+/1Ø$

The 2 polunomials are essentially the same numbers of the previous example, if we put x=1Ø. As such in each step we are essentially collecting the coefficients of different powers of x (which are same as different place values). The produce may be obtained starting from left hand side or right hand side.

Left to Right Computations:

Example 6: Multiply

	1	Ø	2	3
x	2	1	Ø	2

1/2/4/Ø/3/4/6 = 2 1 5 Ø 3 4 6

Start from left hand column

1. 1 / 2 OR 1 x 2 = 2

2. 1 Ø / 2 1 OR 1x1 + 2xØ = 1

3. 1 Ø 2 / 2 1 Ø OR 1 x Ø + Øx1 + 2 x 2 = 4

4. 1 Ø 2 3 / 2 1 0 2 OR 1x2 +2x3 +ØxØ + 1x2 = 1Ø

5. 0 2 3 / 1 Ø 2 OR Øx2 + 3x1 + 2xØ = 3

6. 2 3 / Ø 2 OR 2x2 + Øx3 = 4

7. 3 / 2 OR 3 x 2 = 6

Left to right working is very convenient and useful for practical working, where we do not need the total result.

THE VINCULUM;

We all find that operation of digits 9,8,7 and 6 requires slight more effort. In Vedic Maths we also use Rekhanks (digits with a bar). As such we can always use the complement (Purak) of bigger digits, and thus always work with only small digits (Ø,1,2,3,4 & 5). The use of Rekhanks (bar digits) avoids the large digits., increases the appearance of Ø and 1 and the number often partially or wholly cancel themselves.

9	= 1Ø	− 1	= 1Ø + $\bar{1}$	= 1 $\bar{1}$
18	= 2Ø	− 2	= 2Ø + $\bar{2}$	= 2 $\bar{2}$
29	= 3Ø	− 1	= 3Ø + $\bar{1}$	= 3 $\bar{1}$
99	= 1ØØ	− 1	= 1ØØ + $\bar{1}$	= 1 Ø $\bar{1}$
79	= 1ØØ	−21	= 1ØØ + $\overline{21}$	= 1 $\bar{2}$ $\bar{1}$

So we may use 1 Ø $\bar{1}$ instead of 99, which is far more easier, quicker and simpler. For obtaining the Vinculum numbers, having both positive and negative digits, we make use of the first 2 sutras.

1. We get the complement of bigger digits by using the Nikhilam Navatascaramam Dasatah (All from 9 and last from 1Ø) and write them as Rekhanks (bar digits).

Note: While using Nikhilam sutra we always start from the left hand side.

contd..

2. We use the first Vedic Sutra Ekadhikena Purvena (one more than the previous one) and take Ekadhika (one more) of the previous digit just before the bigger digit/digits), whose compliments have been taken.

Frequently, we may have more than 1 group of big digits, we shall convert each group separately into Vinculum.

$$9 \quad 2 \quad 8 \quad = \quad \bar{1} \; 1 \quad 3 \quad \bar{2}$$

$$389 \quad 1 \quad 9 \quad 2 \quad = \quad 4\,\bar{1}\,\bar{1}\,2\,\bar{1}\,2 \quad \text{etc}$$

In 928 we have two blocks of big digits 9 & 8 separately. Purak of 9 is 1 and Ekadhika of Ø (before it) is 1, as such we get 1 $\bar{1}$ as th vinculum form of the left block. Further, Purak of 8 is 2 and Ekadhika of the previous digit 3 is 3 + 1 = 4. Hence we get 1 $\bar{1}$ 3 $\bar{2}$.

Similarly 89 and 9 are the two large digit blocks in second number 389192, as we would not like to take the compliments of smaller digits (as the compliments shall be bigger than 5, and generally it shall not add to our convenience of working). The Purak of 89 is $\bar{1}\,\bar{1}$ and Ekadhika of the previous digit 3 is 4., and Purak of 9 is $\bar{1}$ and Ekadhika of the previous 1 is 2. Please note that we do not change last 2.

contd..

It is interesting to learn that even the normalisation of Vinculum number is done by using the Nikhilam sūtra alongwith the Ekanyūna Purvena sutra number 14 (One less than the previous one) We take Ekanyunena of the digit before the bar digits etc. Vedic maths also provides a quick check method using the Beejank (Ref.2,3,_6).

The following example shall clearly show the ease and simplicity provided by Vinculum approach of VM.

Example 7: Product of 98 x 88

9 8   1. Vertical: (R) 8 x 8 = 64

8 8   2. Cross : 9x8 + 8x8 = 72 + 64=136

72/6/4   3. Vertical: (L) 9x8 = 72

13/6/

8 6 2 4 = 8624

If we use Vinculum numbers using Nikhilam

$$98 = 1\ \emptyset\ \bar{2}$$

and 88 x $1\ \bar{1}\ \bar{2}$

Now $1\ \ \emptyset\ \ \bar{2}$

$1\ \ \bar{1}\ \ \bar{2}$

$1/\bar{1}/\bar{4}/2/4 \quad = \quad 8\ 6\ 24$

1. Vertical produce $\bar{2} \times \bar{2} = 4$
2. Cross Produce $\emptyset \times \bar{2} - \bar{1} \times \bar{2} = 2$
3. Cross & Vertical products

$1\times\bar{2} + \bar{2} \times 1 + \emptyset \times \bar{1} = \bar{4}$

contd...

4. Cross left product $1 \times \bar{1} + 1 \times 0 = \bar{1}$

5. Vertical product $1 \times 1 = 1$

We observe that absolutely no carry over is required and computations are drastically reduced.

Further, we can check the correctness of results at each stage of working (Ref.2,3, & 6).

Discussions:

1. In both of these alternatives, we have a choice to work either from left to right or from right to left.

2. The carry digits can be added in second step or directly.

3. Even left to right working can be directly done in one line, using the special carry down procedure of VM (Ref. 3 & 4).

4. Urdhava method is the general method of multiplication, and is quite close to the present method which in vedic tradition is called 'Go-mutra vidhi' (गो मूत्र विधि).

5. In addition to the several choices of working within the Udhva method, VM also provides several other methods which are very efficient in handling special pattern problems (Ref.1,2 & 3). One such method, which is useful for the product of numbers closer to any base (10,100 1000 etc) or subbase (20, 30,40, 50 etc are subbases of 10) is based on the 'Nikhilam sutra'.

contd..

6. If we wish to comput th product of 98 x 88 by Nikhilam method, we first of all find the deviation of the numbers from the base (or subbase) using, Nikhilam, Navatascaramam Dasatah Sutra which means, 'All from 9 and last from 10. Multiply the deviations to get RHS.

98	-	02
88	-	12
86	/	24

We get 2 x 12 = 24

Left half of the answer is simply any number added to the deviation of the second number (with proper sign). Further we have two more methods to get LHS.

So LHS is 98 - 12 = 86

OR 88 - 02 = 86

Hence 8624 is the desired result (Ref. 1, 2 & 3)

4. LEFT TO RIGHT CALCULATIONS.

In most of the applications it is very useful to obtain two figures of an answer from left to right, i.e to get the most significant digit first, then the next most significant digits and so on.

4.1. ADDITION

Example 1:

7 6 8 1. Addition of left column gives

8 7 4 7 + 8 = 15 = We put 1 as result and carry forward 5.

1/6/4/2

5/3

2. Addition of central column gives $6 + 7 = 13$, this is added to carry digit 5, which is equal to 5∅ at lower level; $13 + 5∅ = 63$. So we put 6 and carry forward 3.

3. Addition of right column given $8 + 4 = 12$, this is added to the carry digit 3, which is equal to 3∅. So we got $12 + 30 = 42$, and write 42.

Example 2:

$$\begin{array}{rrr} 6 & 8 & 5 \\ 7 & 7 & 2 \\ \underline{5} & \underline{7} & \underline{7} \\ 2/ & 0/ & 34 \\ & \bar{2}/ & 2 \\ \hline \end{array}$$

$= 2034$

Addition of left column gives $6 + 7 + 5 = 18$, which we may write as $2\bar{2}$, write 2 and carry forward $\bar{2}$ etc.,

4.2 SUBSTRACTION.

Example 3:

$$\begin{array}{r} 5^{1}8^{1}2^{1}4 \\ -\ 2\ 9\ 5\ 6 \\ \hline 2\ 8\ 6\ 8 \end{array}$$

Subtraction of left colum gives $5 - 2 = 3$, which we further reduce by 1 and carry it forward, $3-1 = 2$ and next digit 8 becomes 18.

Now $18 - 9 = 9$, but we need to carry forward 1, 20 $9 - 1$ is the figure $12 - 5 = 7$, reduced by one $7 - 1 = 6$; and finally $14 - 6 = 8$.

DISCUSSIONS:

1. VM provides us with several other choices of working (Ref. 2,3 & 6) along with cross checking methods.

2. Additional material is presented in later chapters (Refer chapters 13).

3. If Vedic Maths is studied from the beginning, we may never be required to operate numbers larger than 9 in addition and subtraction.

5. COMBINED OPERATIONS OF ARITHMETIC.

Working with matrices and determinants requires a series of additions and subtractions of products. The mental working system of VM provides us with a large number of choices (Ref.3 & 4).

We shall briefly study one method based on the Urdhva sutra.

Example 1: Compute 21 x 32 + 24 x 22

$$\begin{array}{r} 2\ \ 1 \\ \times\ \ 3\ \ 2 \\ \hline \end{array} \quad + \quad \begin{array}{r} 2\ \ 4 \\ 2\ \ 2 \\ \hline \end{array} \quad = \ 12/Ø\ /\ Ø = 12ØØ$$

1. Vertical product of units gives
   1 x 2 + 4 x 2 = 10
2. Cross product gives
   (2x2 + 3x1) + (2x2 + 4x2) = 19 add 1=2Ø(1 of carry ov
3. Vertical product of tens gives
   2x3 + 2x2 = 1Ø add 2(of carryover) = 12

There is no need to write the intermediate steps.
We could also work from left to write.

1. Vertical product of left columns gives 2x3 + 2x2 = 1Ø, write 1 and carry forward Ø

2. Cross products give 19 and Ø

= 19 which we can use as

```
      2  1        2  4
x     3  2  +  x  2  2
      ----        ----
                        = 1 / 2 / Ø Ø = 12ØØ
                           /  Ø  . 1̄
```

19=2 1̄ so we write 2 and carry forward 1̄

3. Vertical product of right columns give

1x2 + 4x2 = 1Ø added carry forward 1̄ as 1̄ Ø̄

1Ø - 1Ø = ØØ

As such we get 12ØØ

Example 2:

```
      3   1        2   2        4   2
x     2   3  + x   4   1    x   3   2
      -----        -----        -----
      -----        -----        -----  = Ø /2/71= 271
                                          /2/7.
```

Left to Right

1. 3 x 2 + 2 x 4 - 4 x 3 = Ø2

2. Write Ø and carry forward 2 and 2Ø

2. (3 x 3 +1x2) +(2x1 + 2 x 4) - (4x2+2x3)
   = 7, And Ø7 + 20 (carry) = 27
   Write 2 and carry forward 7 as 7Ø.

3. 1 x 3 + 2 x 1 - 2 x 2 = 1

   And 1+ 7Ø = 71

   Right to Left

   2/7/1 = 271

6. SQUARING

6.1 INTRODUCTION

We have seen that how the Urdhva method is used for multiplication of two numbers or more than two numbers. We can also use the Urdhva method for squaring. The following examples describe the procedure.

6.2 LEFT TO RIGHT

We can do the computations from left hand side. But generally we would need to do this in two steps as the carry over digits shall be added in the second step. However, if we adopt the unique carryover approach of Vedic maths system, to the lower side and also use the suitable, alternative form of results at each stage we can get the total result in a single step. Here we keep the 10 th level digit at the same place and shift the unit digit to the R.H.S. which then assumes 10 times its value.

6.2.1 Illustration 1: Square 23

(i) Squaring from left to right (Ref. 2 Chap.8)

```
  2 3
  2 3
 ----
 4 2 9
   1      =  529
```

STAGE I:

Step 1: $RL_1 = 2*2 = 4$

Step 2: $RL_2 = 2*3 + 2*3 = 12$

STAGE II: We start the elimination from L.H.S.

Step 3: $RL_3 = 3 * 3 = 9$

Step 4: Add the carryover digits.

(ii) Square from left to right in one line.

```
     2 3
     2 3
   -----
  ^/5/2/9    =   529
    4 2
```

STAGE I:

Step 1: $RL_1 = 2 * 2 = Ø4$ we retain Ø at this level and shift the unit place digit (here 4) to the R.H.S. which naturally gets its value ten times increased (4*10 = 40)

Step 2: $RL_2 = 2*3 + 2*3 = 12$. To this we add the carried down number(from L.H.S.) 4Ø. Therefore 12 + 4Ø = 52 and out of this number we keep up tenth place digit (5) at this level, and once gain shift the unit place digit (2) to R.H.S., which gets its value increased ten times (2*1Ø = 2Ø)

STAGE II

Since we have already completed the operation of all columns (here only 2) with the left most column, We can start the second stage of elimination from L.H.S. and simplification of the remaining columns.

Step 3: $RL_3 = 3 * 3 = 9$. To this we add the carried down number from (L.H.S.) 2Ø + 9 = 29. Since this is the last step, therefore the number 29 is written in the right most space.

NOTES:

(1) Proceeding like this we have obtained the results in one line.

(2) Since all the digits are small, there was no need to use the vinculum form of the result obtained in different steps. But when we are handling bigger digits we shall find it more convenient and efficient to use the vinculum form of the intermediate results, as has been illustrated in the next example.

(3) We should note that in this method we shall always have two digits in the right hand most space. Even if we get a single digit value in last step it shall be written with a zero prefixed on its L.H.S.

6.2.2 Illustration 2: Square 39.

(i) Squaring from left to right (Ref. Pushp-2, Chap. 8)

39

39

---

9 4 1

58 = 1521

STAGE I:

Step 1: $RL_1 = 3 * 3 = 9$

Step 2: $RL_2 = 3 * 9 + 3*9 = 54$

STAGE II: We start the elimination from L.H.S.

Step 3: $RL_3 = 9 \times 9 = 81$

Step 4: Add the carry digits to get the final result.

(ii) Squaring from left to right in one line.

STAGE I:

Step 1: $RL_1 = 3 * 3 = 9$

$$\begin{array}{r} 3\ 9 \\ \underline{3\ 9} \\ 1/5/21 \\ 1\quad 6 = 1521 \end{array}$$

but it will be more convenient to write this in the vinculum form,

So 9 = 1 1, we retain 1 at this level and shift the unit place digit (1) (read as Rekhanak one) to the R.H.S. which gets its value increased 10 times, Hence 1 x 10=10. As otherwise the cumulative total shall consist of three digits and the hundredth place digit shall spil over the thousand level (90 + 27 + 27=144).

Step 2: RL_2 = 3*9 + 3*9 = 54. To this we add the carried down number from L.H.S.(-10). Therefore 54+(-10)=44, Once again if we carry down the unit place digit (4) we shall end up in three digits at the next step, therefore it shall be convenient to use the vinculum form of this value i.e. 44-56 we retain the tenth level digit (5), and shift the unit place digit (6) to the R.H.S. which naturally gets its value increased by 10 times, So 6*10=60-.

STAGE II

Step 3: RL_3 = 9*9=81. To this we add the carried down number from L.H.S. (Here 60) So 81 + (-60)=21 and since it is the last simiplification we keep two digits at the right most space.

6.3 SQUARING DWANDWA YOGA

6.3.1. INTRODUCTION

We have just learnt the Urdhva method of squarring. Further, for very quick squarring of some particular type of

of numbers which, are near to the base or subbase, using Yavadunam sutra Ref : Pushp 1 and Pushp 5. There is another method for squaring, using Dwandwa yoga upsutra (or the Duplex combination). This sutra can be used in two ways, one is for squaring and second is the cross multiplication.

6.3.2 The Procedure:

Let us learn the method for computing duplexes of different levels.

Level 1: If we have a one digit number, to find the duplex (D) of one digit number we simply sqauar. the digit.

For example 3, $D = 3^2 = 9$.

Level 2: For two digit number we find the duplexes of the digits as double the product of two digits, for example

35, $D = 2(3 \times 5) = 2(15) = 30$

Level 3: For three digit number we find the duplex as the sum of the square of the middle digit and doublt the product of the first and third digits.

For 3 2 7; $D = 2 \times 3 \times 7 + 2^2 = 42 + 4 = 46$

Level 4: For four digit number we find the duplex as the sum of double the product of the first and fourth digits, and, double the product of the second and third digits.

For 1 4 5 6; $D = 2 \times 1 \times 6 + 2 \times 4 \times 5 = 12 + 40 = 52$

Level 5: We extend the same procedure for finding the duplexes of five digit number

For 1 2 3 4 6, D = $2 \times 1 \times 6 + 2 \times 2 \times 4 + 3^2$

$= 12 + 16 + 9 = 37$

Similarly we can extend it even for the larger digit numbers. We can clearly see that we obtain duplex straight forward application of Urdhva Sutra.

6.3.3. Squaring.

The following illustration shall explain the steps.

Illustration 3: Calculate 53^2

Step 1: First we take the duplex of 1st digit 3, as described in level. 1.

So $3^2 = 9$

i.e. the suqare of the first digit.

$\therefore 53^2 = \; / \quad /9$

Step 2: Now we take the duplex of first and second digits as described in level 2, so double the product of the two digits.

$\therefore$ Duplex of 53 = $2 \times 5 \times 3 = 3Ø$

Therefore $53^2 = \; / \; 3 \quad Ø \; /9$

Step 3: Further we take the duplex of 2nd digit only, as in level

1. So Square of 2nd digit is

$5^2 = 25$

$53^2 = 25/ \; Ø \; /9$

Step 4: Add the carry digits.

$53^2 = 25/\not{0}/9 = 2\not{0}\not{0}9$

- Hence $53^2 = 2\not{0}09$

NOTES: We can easily start the process of squaring from left hand side.

Illustration 4: Calculate 8342^2

STAGE 1

Step 1: Duplex of first digit is $2^2 = 4$

$(8\ 3\ 4\ 2)^2 = / / / / / / / / / 4$

Step 2: Duplex of 1st and 2nd digit (level 2)

So for 42: D = 2 x 4 x 2 = 16

$\therefore (8342)^2 = / / / / / 6/ 4$
1

Step 3: Duplex of 1st 2nd and 3rd digits (level 3).

For 342; $D = 2 \times 3 \times 2 + 4^2 = 12 + 16 = 28$

$\therefore (8342)^2 = / / / / / 8 / 6 / 4$
2 1

Step 4: Duplex of the first four digits (level 4)

For 8342: D = 2 x 8 x 2 + 2 x 3 x 4 = 32+24 = 56

$\therefore (8342)^2 = / / / 6 / 8 / 6 / 4$
5 2 1

STAGE II : Now we start the elimination process.

Step 5: Duplex of 2nd, 3rd and 4th digits (level 3) leaving 1st digit.

contd..

For 834; $D = 2 \times 8 \times 4 + 3^2 = 64 + 9 = 73$

$$\therefore (8342)^2 = \;/\;/\;3/\;6/\;8/\;6/\;4$$
$$\quad\quad 7\;\;5\;\;2\;\;1$$

Step 6:

Duplex of 3rd and 4th digits i.e. 83; $2 \times 8 \times 3 = 48$

$$\therefore (8342)^2 = 8/\;3/\;6/\;8/\;6/\;4$$
$$\quad\quad 4\;\;7\;\;5\;\;2\;\;1$$

Step 7: Duplex of 4th digit only i.e. $8^2 = 64$

Step 8: Adding the carry digits we get sqauar of the number

$$(8342)^2 = 64/8/3/6/8/6/4$$
$$\quad\quad 4\;\;7\;\;5\;\;2\;\;1$$
$$= 69588964$$

Hence $(8342)^2 = 69588964$.

6.3.4 VINCULUM FORM: If we have bigger digits, converting to vinculum form shall reduce the computational effort.

Illustration 5: $2199^2 = 220\bar{1}^2$

$= 4/8/4/\bar{4}/\bar{4}/0/1 = 4835601$

ONE LINE POSITIVE RESULT:

We have just seen how convenient it becomes to use the vinculum form of numbers. But it has the apparent deficiency of not giving the answer in one line.

As some of the digits of the result are obtained in vinculum form. However, if we are interested in getting the positive result in one line, we have just to convert the bar digit into its complementary digits and carry additional $\bar{1}$ to the higher place as already explained in Pushp-2 (2nd Editition). Complete details are available in the correspondence course.

<u>Illustration 6:</u> Square 2199. Let us do this again.

$$2199^2 = 22\bar{Ø}1^2 = 4/8/3/\underset{\bar{1}}{5}/\underset{\bar{1}}{6}/Ø/1 = 4835601$$

In step 3 we obtained the duplex as 2*2*1 = Ø*Ø = 4 which is written as $1\bar{6}$ and is shifted to the higher level and added to the next duplex number. In step 4 the dupliex obtained is $\bar{4}$ this added to carryover. $\bar{1}$ gives us $\bar{5}$ which is used in the alternative form i.e. $1\bar{5}$ etc.,

DISCUSSIONS:

Proceeding on the same lines, it is also possible to compute the results., even from left to right, while using the vinculum form of numbers, in one line. Here we shall shift the unit digit to the lower level as explained in Pushp-2 and correspondence course.

There is yet another method of squaring. This upsutra reads as follows Yavadunam Tavadunikrtya Vargarca Yojayet (यावदूनं तावदूनीकृत्य वर्गं च योजयेत्) which means whatever the extent of its deficiency lessen it to the same extent and set up the square of the deficiency (Refer Pushp in 1 Chapter 9).

Illustration 7: Calculate the square of the number 98.

Step 1: First we select the base nearest to the number. Here 1ØØ is the base.

Step 2: We divide the result in two parts by drawing a slash

98^2 = L.H.S./R.H.S.

Step 3: The deficiency of the number from the base is (1ØØ - 98 = Ø2) = Ø2

Step 4: Now we substract this deficiency from the number to get the L.H.S. as 98-Ø2 = 96

So 98^2 = 96/R.H.S.

Step 5: For R.H.S. result we take the square of the deficiency. So the sqauer of Ø2is Ø4. Hence the result is 98^2 = 96/Ø4 = 96Ø4. As there are 2 zeroes in the base (1ØØ) we must have 2 digits on R.H.S.

7. DIVISION

7.1 INTRODUCTION

In AVM Pushp-2 we have studied the use of three sutras, namely Nikhilam (No.2), Paravartya (No.4) and Urdhva (No.3), combined with Upsutra Dhwajank for division of numbers. Each one of these is most conveniently and efficiently applicable in particular problems. Their applicability and working efficiency all the more increases when we freely use the Vedic Vinculum approach. We shall consider some additional working details in the section.

contd..

7.2 STRAIGHT DIVISION

In Nikhilam method of division, we use the Nikhilam sutra "Nikhilam Navatashcaramam Dashatah" (निखिलं नवतश्चरमं दशतः) too obtain the modified division (MD) as a first step. Further, we use the MD for subsequent operations (Refer Chapter 9, Pushp-2). Further, it has been shown that it is much more convenient to use the Vinculum numbers in place of bigger digits.

Illustration 1: Here we use the Dhwajank upsutra which means "on the top of the flag". Further, when the flag digits are more than 1 we also use the Urdhva Sutra (No.3) the whole procedure is purely mental and after practice division can be done straight away in one line.

Divide 518 by 23

23) 518

Step 1: We write it as follows

F.D.

3) 518

(operator)2

Here 3 is the flag digit.

Step 2: We put dots for the remainder portion, equal to the number of flag digits.

3) 51 : 8 :

2

Step 3: (i) Divide 5 by 2 we get 2 as first quotient digit and 1 as the remainder and place it before the next digit as

3) 51 : 8 :

2 1

2

So our next gross dividend is 11. Net dividend (ND)= 31 - 3 x 2 = 5.

Step 4: Divide net dividend (i.e.5) by 2, we get 2 as the 2nd quotient digit and 1 as remainder and place it before 8 as,

$$\begin{array}{r|l l}
3 & 51 & : \ 8 \\
2 & \underline{\ \ 11} & \\
 & \underline{22} &
\end{array}$$

Net gross remainder is 18, as we have reached the remainder dots.

(ii) From 18 subtract

$$(3x2) \quad = -\underline{\ 6} \\ \underline{12}$$ So 12 is our final remainder

$$\begin{array}{r|l l}
3 & 51 & : \ 8 \\
2 & \underline{\ \ 11} & \\
 & \underline{22 \ : \ 12} &
\end{array}$$

Q = 22

R = 12

Illustration 2: In the previous example we have seen that out of the two digits of divisor we are actually dividing by using one digit and the flag digit is being used for simple readjustment of the gross dividend. Now let us take an example of division by the three digit number.

Divide 7142695 by 824.

Here the divisor is of 3 digits. In this case we place the last two digits of divisor as flag digits and adopt a slightly different modus operandi on Urdhva Tiryak Lines as follows:

24 √71462 : 95:

8

The initial two steps are done already explained, with last 2 digits marked off.

Step 1: (i) divide 71 by 8 we get 8 and remainder is 7.

24 √71426 : 95

8 7

8

So 74 is the gross dividend.

(ii) From 74 we subtract the product of the 1st digit of flag number (i.e.2) and the 1st quotient digit (i.e.8) So we get 78 − (8x2) = 74 − 16 = 58. So 58 is our next net dividend(ND).

The subsequent steps have a slight modifications. We ust the Urdhva sutra for adjustment of our gross dividend.

Step 2: (i) Divide 58 by 8 we get 6 and 10 as remainder,

24 √714 26 : 95 :

8 710

86

(ii) From 102 we subtract, (By Urdhva tiryak rule) gross products of two flag digits (24) and two quotient digits just obtained (86) i.e.

$$\begin{array}{l} 24 \\ \times \\ 86 \end{array} \begin{array}{l} = 12 + 32 \\ = 44 \end{array}$$

So our remainder is

$$\begin{array}{r} 102 \\ -\ 44 \\ \hline 58 \end{array}$$

(which is our next net dividend (ND).

Step 3: (i) On dividing 58 by 9 we get 6 and 1Ø as remainder.

Our next gross dividend is 1Ø6

24/	7	14	2	6	:	95 :
8	7	1Ø	1Ø			
	8	6	6			

(ii) From 1Ø6 we subtract, By Urdhva Tiryak rule gross products of two flag digit (24) and the last 2 quotient digits (66) i.e.

$$\begin{matrix}24\\ \times\\ 66\end{matrix} \quad 12 + 24 = 36$$

and the remainder is 1Ø6

−36

7Ø which is our next net dividend.

Step 4: (i) Dividing 7Ø by 8 we get 8 and 6 as remainder.

Our first gross remainder is 69, as we have reached the remainder portion.

24 /	7	1	4	2	6	:	95
8		7	10	10	6		
		8	6	6	8		

(ii) From 69 we deduct by Urdhva Tiryak rule, gross product of two flag digits (24) and last 2 quotient digits 68 i.e.

$$\begin{matrix}24\\ \times\\ 68\end{matrix} = 16 + 24 = 4Ø$$

The next level remainder is 69

− 4Ø

29 and 295 is therefore the gross

R.

This has to be further corrected by the cross multiplication of 24 with 80 = 2*Ø+8*4 = 32.

Net remainder R = 295-32 = 263.

This can also be obtained directly by obtaining the correction using the first stage of computations of Urdhva method.

24 = 432: Therefore net R = 695 - 432 = 263

x 68

32

43

Illustration 3: If the number of digits in the divisor are large and we are interested to obtain only a limited number of digits of the quotient (For example when we are computing percentage of population etc) It is interesting to observe that many of the digits of the divisor as well as the dividend may not be touched even once and still we can get the exact required value by using this Vedic general method of division based on the upsutra Dhwajank and sutra Urdhva Tiryakbhyam (No.3).

What is the percentage value of 432461 of 32738678.

The desired percentage

=(432461/32738678)*1ØØ

2738678 | 4: 3 2 4 6 1 Ø Ø

1 2 3 Ø

1. 3 2 2

= 1.32%

contd..

Step 2: D = 13 - 2*1 = 11 .

Step 3; D = 22-(3*2 + 7*1) = 9

Step 4: D = 34-(1*3 + 7*3 + 2*2) = 6

Notes: 1. Step 4 may be done simply to confirm the order of the next digit which if big may be suitably rounded off.

2. In step 3 we may be tempted to take 3 as the quotient and then in step 4 we shall have

D = ∅4 - (1*3 + 3*7 + 3*2)

= - 26. Therefore Q4 = 8

and % = 1.3.3.$\bar{8}$ = 1.32 Ans.

3. However in case we are interested to compute further digits the simple Dhwajank procedure is continued

Step 5: D= ∅6 - (2*2 + 7*2 + 3*3 + 1*8)

= - 29

Therefore Q5 = -9 and R = $\bar{2}$ etc

Similarly we can continue further.

COMMENTS:

This is another wonderful and unique feature of Vedic division method as we are required to do only the bare minimum essential amount of computations.

This is perfectly in tuned with the Vedic tradition where a human being, the roof and crown of Gods' creation, is required to follow the path of divine bliss and fulfilment. The whole Vedic approach is towards poised and balanced minimization of efforts and maximization of results

8. SQUARE ROOTING.

8.1 INTRODUCTION

To find the square root of the number using the vedic sutra is very simple, easy and interesting. The sub sutra named Dwandwa Yoga (द्वन्द्व योग) or duplex process has two meanings. First one is by squaring and the second one is cross multiplication. This sutra is very easy to apply. Using this sutra we can also find the squares of the numbers. In previous sections we have seen different methods of squaring.

<u>8.1 Illustration 1:</u> Find the square root of 4489.

Step 1: First we make the pairs of two digits each from the given number R.H.S. to L.H.S. and write as

44 : 89:

The first pair of digits from L.H.S. is written outside the dots and rest of the pairs are written in between the dots.

Step 2: Now we find the squares of the first nine natural numbers as 1,4,9,16,25,36,49,64,81 and choose one of them which is most suitable and lesser than the first pair.

(i) In this case square of 6 is suitable. So 6 is the first quotient digit. After this step we find the remaining part of the square root using straight division process. So substract the square of 6 (i.e. 36) from 44 we get (44-36 = 8) 8 as a remainder which is prefixed as the first digit of the next pair (i.e. 8) Hence 88 is the dividend.

contd..

(ii) For finding th divis r w doubl th first quoti nt digit (i.e. $2 \times 6 = 12$) obtain d in step 2(i) and write as follows:

$$\begin{array}{l} 44 : \quad 89 \\ \quad\; 8 \\ \underline{12 \qquad\quad} \\ \;: 6 : \end{array}$$

So 12 is our divisor. Th ntir division is done by 12. Step 3: (i) Now divide 88 by 12 we get 7 as a quotient and 4 as the remainder, which is prefixdd to the next digit 9. So 49 is our next gross divid nd.

$$\begin{array}{l} \quad\;\; 44 : \quad 8 \;\; 9 : \\ 12 \; \underline{\qquad 8 \;\; 4 \qquad} \\ \qquad\;\; : \; 6 : 7 : \end{array}$$

(ii) Now subtract th squ of th quoti nt digit (i. .7^2 = obtained in (S p 3(i) for th gro s dividend 49 we get (49-49) Ø as quotient and Ø as r maind r. Hence th squar root is 67, which is an exact squar .

DISCUSSION:

(1) We hav just s en h w asy it is to find the square root of any big numb r.

(2) If a numb r contains n digits.

(i) If n is odd than th squar root of a numb r shall h ve $\frac{n \pm 1}{2}$ numb r of digits.

(ii) If n is even than th squar root of numb r shall have digits.

(iii) If the number is of pure decimal then the number of digits in the square, is double that in the square root.

(iv) If the square root is not an exact number, the decimal shall be put after the number of digits equal to the number of pairs formed.

8.2 Illustration 2: Find the square root of 17.

```
17 : 0 0 0 0 0 0 0 0
 8 : 1 2 3 2 2 6 7 3
     ---------------
   : 4 1 2 3 1 0 5 6
     ---------------
```

the net dividend at each stage is

10

$20 - 1^2 = 20 - 1 = 19$

$30 - 2 \times (1x2) = 30 - 4 = 26$

$20 - (2x\ (3x1) + 2^2) = 20 - 10 = 10$

$20 - (2 \times (1x1) + 2 \times (2x3) = 20 - 14 = 6$

and so on Hence, the square root of 17 is, 4.1231056 which is not an exact root.

8.3 Illustration 3: Find the Square root of 1 2 4 3 5 9

```
: 1 2 : 4 3 . 5 9 0  0
        3 4 / 6 9 11 16
        ---------------
  6:3:5.26    4  8 --
        ---------------
```

Hence the square root is 35.2647 etc.

VINCULUM APPROACH

Let us solve this illustration again using Vinculum

```
: 12 :  4  3   5 9  0 0 0
  6     34   0   1  2 0 0
  -----------------------
   3:5 3  4     5  1    6   =  35.26484 etc.
  -----------------------
```

Further details are available in AVM Pushp 3.

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CHAPTER 2

PART II

2.1 ALGEBRA OF COMPLEX NUMBERS USING SUTRAS OF VEDIC MATHEMATICS.

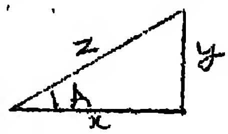
The sutras of Vedic Mathematics have varied applications in different branches of mathematics. Several applications have been presented in Teacher's Guide for books of Class IX, X and XII.

Even the algebra of complex numbers can be very conveniently done by using the Vedic Tribhujank approach. The Tribujank approach provides very convenient and direct solution of varied problems of 2-D and 3-D coordinate geometry as well as trigonometry, both plain and spherical (to be discussed in later chapters). Further details are available elsewhere (R [illegible] 3 & 5).

Let us introduce ourselves to the VM Tribhujank approach.

The ancient Vedic Rishis had complete knowledge and understanding of geometry also. This is very well accepted by all modern scholars. Herein we introduce ourselves to a very interesting, useful and thought provoking facet of ancient knowledge. A set of three numbers x, y, and z which satisfy the condition $x^2 - y^2 = z^2$ is called Tribhujank (त्रिभुजांक). This word consists of three parts, Tri (pronounced as Tre) means three; bhuja means the side; ank means the numbers. Therefore Tribhujank stands for a set of three numbers which essentially represent the three sides of a right angled triangle, having x as the base, y as the height and z as hypotenuse. Tribhujank

written in this sequence essentially represents the angle (A) between the sides x and z.



The so called Pythagorus theorem was extensively used by the ancient Vedic people a few thousand years before the Pythagorus period. The following four formulae from Katyayan's sulbha sutras clearly demonstrate their perfect understanding of this simple and important fact of geometry.

1. सम चतुरस्रस्य अक्ष्णया: द्विकरणी ।
2. द्वि प्रमाणा चतु: करणी, त्रि: प्रमाणा नवकरणी, चतु: प्रमाणा षोडश करणी
3. अर्धप्रमाणेन पाद प्रमाणम् विधीयते ।
4. तृतीयेन नवमी कला ।

Meaning thereby

1. The rope measuring the diagonal of a square can help in constructing the square twice in area as the first.

2. With the help of a rope twice the length of a side of a square, you can construct another of four times in area, thrice in length-nine times and four times in length-sixteen times.

3. Half the length of rope will yield quarter area.

4. One third length of rope will yield one ninth area.

Even one old Babylonian tablet (Plimpton collection of Columbia University, New York) which dates from 1900–1600 B.C. presents a set of fifteen Tribhujanks, decreasing in steps of 1°.

The set of numbers 3,4,5 represents one Tribhujank as $3^2+4^2 = 5^2$. If we know the hypotenuse and any of the other two sides, the Vedic Sutra (सङ्कलनव्यवकलनाभ्याम्) Sankalana Vyavakalanabhyam (No.7) can be used to compute third side very conveniently. The sutra means by addition and subtraction.

Example 1: If 13 is the hypotenuse (z) and 12 is the base (x) of a Tribhujank and the third side, height is y then

$$y^2 = z^2 - x^2$$

$$= (z + x)\ (z - x)$$

Therefore $y^2 = (13+12)\ (13-12)$

$$Y^2 = 25$$

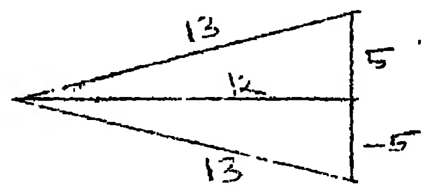
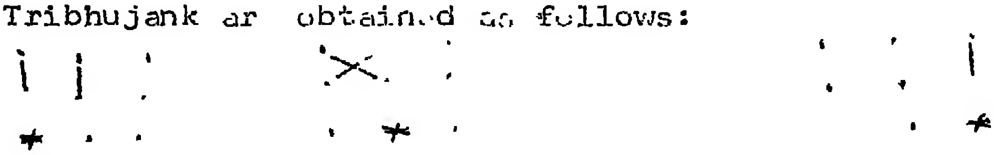


Fig. 1

and y= 5. In general y can also be -5 (See Fig. 1).

ADDITION AND SUBTRACTION OF TRIBHUJANK.

Both of these operations can be easily done by direct application of the Urdhva sutra (No.3) which simply means vertically and crosswise; only the specific procedural details have to be noted. The three elements of the new Tribhujank are obtained as follows:



1st element 2nd element 3rd element.

If(A) x_1, y_1, z_1 and

(B) $x_2\ y_2\ z_2$ are two Tribhujanks containing angle A & B respectively. Then the Tribhujank

contd..

contained Angles A − B and A + B are

A	x_1	y_1	z_1
B	x_2	y_2	z_2
A−B	$(x_1 x_2 + y_1 y_2)$	$(x_2 y_1 - x_1 y_2)$	$(z_1 z_2)$
A + B	$(x_1 x_2 - y_1 y_2)$	$(x_2 y_1 + x_1 y_2)$	$(z_1 z_2)$

Let us consider one example.

Example 2: Find Tribhujank of Angle A−B and A+B, if

A	4	3	5
B	12	5	13
A−B	(4x12+3x5)	(12x3−4x5)	(5x13)
	63	16	65

Similarly;	(4x12−3x5)	(12x3+4x5)	(5x13)
A + B	33	56	65

2.2. Operation of complex Numbers:

(a) ADDITION AND SUBTRACTION:

Addition and subtraction of complex numbers is done by using the Vedic sutra 'First by first and last by last' i.e. while adding complex numbers, the real part is added to the real and coefficient of 'i' (complex part) is added to the complex part, same is true for subtraction of two complex numbers.

(b) MULTIPLICATION OF COMPLEX NUMBERS:

The multiplication of complex numbers can be done just as we add the Tribhujanks.

Example 3: Multiply $(4+3i)\ (12 + 5i)$

We consider the coefficients as the two elements of Tribhujank and carryout the addition.

$$\begin{array}{ccc} 4 & + & 3i \\ \underline{12} & \underline{+} & \underline{5i} \end{array}$$

$$(4.12 - 3.5) + (3.12 + 4.5)\ i$$

$= 33 + 56i$ is the resulting complex numbers.

(c) DIVISION OF COMPLEX NUMBERS

The complex numbers can be divided like the subtraction of Tribhujanks. We must also divide by the square of the third element of the Tribhujank.

Example 4: $\dfrac{4 + 3i}{12 + 5i}$

So

$$\begin{array}{ccc} & 4 + 3i & \\ & 12 + 5i & 13 \\ \hline \end{array}$$

$$((4.12 + 3.5) + (3.12 - 4.5)\ i\ ) / 169$$

$= \dfrac{63 + 16i}{169}$ is the resulting complex number.

Example 5: Compute $(i-3)/(2i+ 1)$

WE got

$$\begin{array}{ccc} -3 & + & i \\ 1 & + & 2i \\ \hline \end{array}$$

5

$$\{(-3.1 + 1.2) + (1.1 - (-3)\ .2)\ i\}/5$$

$= (-1 + 7i)/5$ is the resulting complex Number.

(d) SQUAREROOT OF A COMPLEX NUMBER

is simply the half angle Tribhujank of the given elements of the complex number which can be directly computed by using the half angle (Beejank formula). The details are presented in chapter 12 and 13 (Ref. 3 & 5 for complete details).

Example

$$\sqrt{8 + 6i} = 3 + i$$ as 3, 1 are the Beejanks for Tribhujank (8, 6, -)

Similarly, 3, 2 are the Beejanks for Tribhujank (5, 12, -)

2, 1 are the Beejanks for Tribhujank (3, 4, -)

4, 1 are the Beejanks for Tribhujank (15, 8, -)

In general $$\sqrt{x + iy} = \sqrt{\frac{x + r}{2}} + \frac{yi}{\sqrt{2(x+r)}}$$

where (x, y, r) is a tribhujank.

So, $$\sqrt{8 + 6i} = \sqrt{\frac{8 + 10}{2}} + \frac{6i}{\sqrt{2(18)}} \quad (8, 6, 10)$$

$$= 3 + i$$

$$\sqrt{3 + 4i} = \frac{3 + 5}{2} + \frac{4i}{\sqrt{2 \times 8}} \quad (3, 4, 5)$$

$$= 2 + i$$

$$\sqrt{4 + 3i} = \sqrt{\frac{4 + 5}{2}} + \frac{3i}{\sqrt{2(9)}}$$

$$= (3+i)/\sqrt{2}$$

$$\sqrt{12 + 5i} = (5 + i)/\sqrt{2}$$ etc.,

CHAPTER - 3

APPLICATIONS OF VEDIC MATHEMATICS IN THE QUADRATIC EQUATIONS.

We have studied some examples of quadratic equations, and how to form new quadratic equations whose roots has a definite relationship with the roots of the given equation.

Vedic Mathematics sutras provide simple procedures for obtaining directly 4 types of equations. Further, these procedures are applied directly even to the higher order equations, although here we shall be considering mostly the quadratic equations only.

A. To obtain an equation whose roots are opposite in sign to those of a given equation.

We use the fourth Vedic sutra, which says 'Paravartya Yojajet' - meaning 'Transpose and Apply', so we transpose the sign of every alternate term, starting with the second and leaving the constant.

Example 1: $15x^2 - 13x + 2 = \emptyset$ has roots 2/3 and 1/5

$\therefore$ $15x^2 + 13x + 2 = \emptyset$ has roots - 2/3 and -1/5

Example 2:

Similarly $x^2 - 3\emptyset x - 216 = \emptyset$ has roots - 6 and 36

$\therefore$ $x^2 + 3\emptyset x - 216 = \emptyset$ has roots 6 and -36

B. To obtain an equation whose roots are reciprocals of those in a given equation.

We reverse the order of the coefficients.

Example 3: $2x^2 + 5x - 3 = \emptyset$ has roots 1/2, - 3

$\therefore$ $-3x^2 + 5x + 2 = \emptyset$ has roots 2, (-1/3)

Similarly

Example 4: $x^2 - 8x - 9 = \emptyset$ has roots 9, -1

$-9x^2 - 8x + 1 = \emptyset$ has roots $1/9$, -1

The proofs of these simple and direct procedures are available elsewhere (Ref.3,4).

C. To obtain an equation whose roots are all equal to those of a given equation multiplied by a given number.

This case comes under the first Vedic upsutra, 'Proportionately' - 'Anurupyena'.

We multiply the coefficients of the given equation by successive powers of the given number, starting with zero.

Example 5: $2x^2 - 5x - 3 = \emptyset$ has roots $-1/2$, 3

supposing we want an equation with double these roots, i.e. -1, 6.

Multiply the coefficients by 2^0 2^1, 2^2

$\therefore\ 2x^2 - 5 \cdot 2x - 3 \cdot 2^2 = \emptyset$

OR $2x^2 - 10x - 12 = \emptyset$ is the required equation.

Example 6: The Equation $2x^2 - 13x + 15 = \emptyset$ has roots $3/2$, 5, If we want an equation having double the roots, 3, 10

$2x^2 - 13.2x + 15.4 = \emptyset$

OR $2x^2 - 26x + 60 = \emptyset$

OR $x^2 - 13x + 30 = \emptyset$ is the required equation.

Further $2x^2 - 13.4\,x + 15x16 = \emptyset$

OR $2x^2 - 52x + 240 = \emptyset$

OR $x^2 - 26x+12\emptyset = \emptyset$ shall hav 4 tim's th roots i.e. 6.2$\emptyset$

D. To increase or decrease the roots of an equation by a given numb r t.,

We obtain th successiv derivatives of th polynomial, and substitute the transposed value of the given number(−t).

Example 7: Reduce the roots of the equation

$2x^2 - 13x + 15 = \emptyset$ (roots 3/2,5)

by 1

$\therefore -t = -1$

If $y = 2x^2 - 13x + 15$ $y(-1) = 2 + 13 + 15 = 3\emptyset$

$y' = 4x - 13$ $y'(-1) = -17$

Hence Equation is

$$2x^2 - 17x + 3\emptyset = \emptyset$$

Exampl 8: $x^2 - 8x - 9 = \emptyset$ has roots 9, −1.

to find an equation whose roots are 3 more than these roots

$\therefore -t = -3$ Th equation should have roots 12,2

$y = x^2 - 8x - 9$ $y(-3) = 9+24-9 = 24$

$y' = 2x - 8$ $y'(-3) = -6-8=-14$

As such the desired equation is

$$x^2 - 14x + 24 = \emptyset$$

Example 9: Equation $x^2 - 5x - 6 = \emptyset$ has roots − 1,6

We wish to hav an equation whose roots are 3 less.

So the roots should b − 4,3,

$-t = 3$

$$y = x^2 - 5x - 6 \qquad \therefore\ y(3) = 9-15-6 = -12$$

$$y' = 2x - 5 \qquad \therefore\ y'(3) = 6-5 = 1$$

As such the desired equation is

$$x^2 + 1x - 12 = 0$$

Discussions:

The roots of the quadratic equations which can be factorised, can be directly obtained. Vedic Mathematics provides easy methods for factorising, which have been presented in the teachers manuals for Class IX and X. Further, details are available elsewhere (Ref. 1,3,4).

CHAPTER ...4

SEQUENCES AND SERIES

Vodic Methods: Frequently we are tempted to wrongly think that mathematical topics like Arithmetic Progression, Geometric Progression, Permutations and Combinations, and Binomial Theorem etc. are recent contributions of West. This is a very wrong feeling as detailed discussions and procedures of these and many other topics are available in the ancient texts of Bhaskaracharya and Sridharacharya, Bhaskaracharya Leelawati discusses these topics in details alongwith a number of interesting examples in Chapter 4 and 5. We find the descriptions of AP even in the Vedic texts. Further for working out the problems of these topics the simple and easy multiple choice methods of vedic mathematics can help us to perform the computations with ease and simplicity. Vedic methods of computations as outlined in the chapter of introduction can be directly used. Further details are available elsewhere (Ref. 2 & 3).

COMPUTING

Computer Arithmetic - Vedic Mathematics : It is interesting to note that the method of subtraction using compliments which we learn in this topic is exactly in time with the Vedic method. There is a perfect similarity between the two (Ref.2 & 3).

Vedic methods have been found to be easier to learn and delightful to use. After regular systematic practice of a few weeks we can do most of the computations directly. As we use the vedic procedures of mental working we shall save time and effort.

CHAPTER 9

COORDINATE GEOMETRY

Applications of Vedic Mathematics in Problems of Straight Line.

1. INTRODUCTION

The application of Vedic sutras help us to solve a number of problems directly. Further, the use of Vedic Tribhujank approach drastically reduces the number of formulae required in working out problems of Co-ordinate Geometry (both 2-D and 3-D) and Trigonometry (both plane and spherical). The Tribhujank approach of Vedic Mathematics has been briefly introduced in the second chapter on Algebra of complex numbers.

A few examples are presented here and also later on in Trigonometry, Illustrating the efficiency of the Vedic methods. For better understanding the Chapters of trigonometry may be studied before these illustrations.

APPLICATIONS OF VEDIC MATHEMATICS IN STRAIGHT LINE PROBLEMS

9.3 EQUATION OF THE LINE

The euqation of a line passing through two points can be written directly by using sutra Paravartaya Yojyet (No.4) by a mere casual look at the coordinates. The constant term can also be obtained by using the upsutra Adyam Antyam and Madhyam (No.3) which means the product of the means minus (-) the product of the extremes (Ref. 1).

Example 1: Find the equation of a line passing thtough two points (8,9) and (4,5)

$$\begin{array}{cc} 8 & 9 \\ 4 & 5 \\ \hline \end{array}$$

$$(8-4)y = (9-5)x + (8*5-9*4)$$

$$\therefore \quad 4Y = 4x + 4$$

$$y = x + 1$$

The coefficients of x and y are the difference of the y and x coordinates respectively and the constant term is the determinant of the coordinates.

Example 2: Find the equation of a line passing through (3,5) and (2,3).

3	5
2	3

$$(3-2)y + (5-3)x + (3*3-2x5)$$

$$\therefore y = 2x - 1.$$

9.6 NORMAL FORM

Vedic Tribhujank approach provides the straight and direct solution to most of the problems. Further, in the Vedic Mathematics approach we do not need many formulae. We had our first introduction to the Tribhujanks in Chapter 2. Let us work out the Tribhjanks of some standard angles before using them to solve problems of straight lines.

Example 3: Find the equation of the line through the point (2.3) which makes an angle of 45^0 to the line y-= x + 1. There are two lines: the Tribhujanks for the given line and 45^{o} are

line	1	1
45^o	1	1
By adding	∅	2
Therefore the line is ∅*y = 2x+C)		
By subtracting:	2	∅

$\therefore$ 2x + C = ∅

$\therefore$ line 2y=C

Since the lines pass through the point (2.3) we get 2x-4 = ∅ and 2y = 6.

We can see that how convenient it is to solve this class of problems by using the Tribhujanks.

<u>Example 4:</u> Find the equation of circle with a centre (2,3) which is tangential to the line y=3x.

We need the radius of the circle, AC, since the circle is tangential to the line y= 3x, the sum of the angles between the centre and the x-axis (S) and the centre and the tangent (F) has a gradient of 3. Therefore, the second angle (S) and hence the radius of the circle can be obtained by finding the difference of the tribhujanks (S + F) and S(see fig.1)

Tangential line (S+F)	1	3	$\sqrt{10}$
Centre (S)	1	3	$\sqrt{13}$
(S+F) − S	11	3	$\sqrt{130}$
OR	$11/\sqrt{10}$	$3/\sqrt{10}$	$\sqrt{13}$

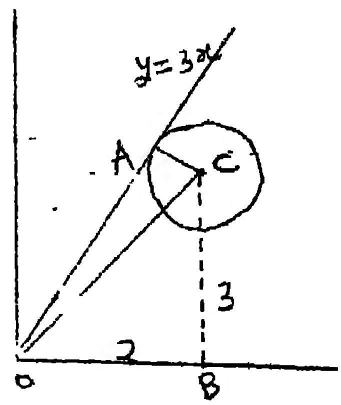


Fig. 1

Since OC $=\sqrt{13}$ $\therefore$ AC $= 3/\sqrt{10}$

Therefore the equation is ..

$$(x-2)^2 + (y-3)^2 = 9/10$$

Since we are interested in the actual lengths therefore the Tribhujank has to be divided by suitable number (here $\sqrt{10}$). Further, frequently we do not require all the three elements of the Tribhujank.

EXAMPLE:5 :

What is the length of perpendicular from the point (4,3) to the line y= 2x.

SOLUTION: If we subtract the Tribhujank for point from the Tribhujank for the line we shall get the Tribhujank for the remaining angle. The middle element of which shall be the desired length of perpendicular (See F.2).

line	1	2	$\sqrt{5}$
Point	4	3	5
Subtracting	10	5	$5\sqrt{5}$

Triangle OPQ = $10/\sqrt{5}$ $5/\sqrt{5}$ 5

Therefore PQ $=\sqrt{5}$

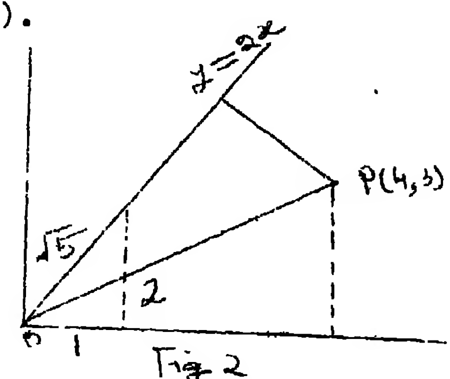


Fig 2

Example 6: Compute the distance of the point (-2, -3) from the line 4y + 3x = 0

line	4	-3	5
point	-2	-3	-
Subtracting	-	18	-

Therefore PQ=18/5 = 3.6

Example 7: Find the equation of the straight line passing through the pair of points (3,4) and (2,-1)

3	4
2	-1

(3-2)y= 4-(-1) x + 3*(-1) - 4*2

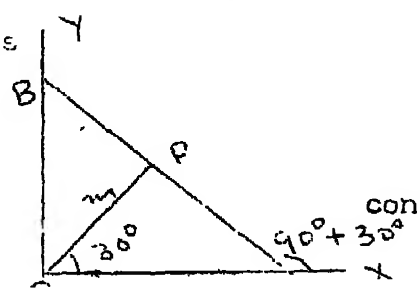
OR y=5x -11 is the desired equation.

Example 8: Find the

{Ex.9.6 p.174 of Pt.II Class XI}

We know that Tribhujank of 30^o is

30^o) $\sqrt{3}$ 1 2



contd..

Therefore co-ordinates of P are $(2\sqrt{3}\ ,\ 2)$

Tribhujanks of x-axis) is	1	0	1
of 120°) is, Adding +	-1	$\sqrt{3}$	2
Tribhujank of BAX) is − 1	-1	$\sqrt{3}$	2

Therefore equation of line APB is

$$-y = \sqrt{3}x + c \quad \text{since it}$$

passes through $P(2\sqrt{3},2)$ $\therefore\ -2 = 6 + C$ OR $C = -8$

Substituting the value of C

Therefore $-y = \sqrt{3x} - 8$

OR $y + \sqrt{3}x = 8$ is the required equation.

<u>EXAMPLE 9:</u> (Take 9.7 Ex.pp 175 of Book)

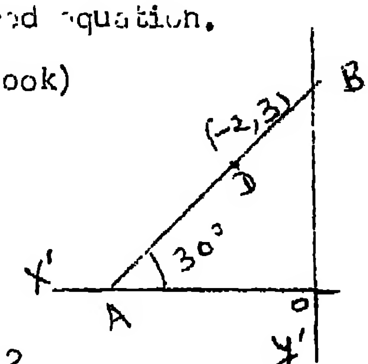


Fig. 4.

Tribhujank for

x-axis)	1	0	1
30°)	$\sqrt{3}$	1	2
Adding + Tribhujank for OAB)	$\sqrt{3}$	1	2.

Therefore equation of line ADB is

$$\sqrt{3}y = x + C$$

since it passes through point D $(-2,3\bar{3})$

$$\therefore\ x - \sqrt{3}y = 2+3\sqrt{3}$$

is the required equation.

Discussions: Further the Vedic rishis had a through knowledge of circles and conics. They had very simple and direct working rules. Only simple illustrations have been presented here due to paucity of space. Further details are available in Sulba sutras and other texts.(Ref.3,8 & 9).

CHAPTER 13

TRIGONOMETRIC FUNCTIONS

Vedic Mathematics Approach

1.1 INTRODUCTION

This branch of mathematics was very well developed in the ancient Bharat. We have several details available in Samskrit literature (Ref.8,9,10) dealing with astronomy etc. In addition to the long form treatment Vedic Mathematics also provides direct and simple approach of Tribhujank for solving most of the problems of Trigonometry (both plain and spherical) as well as coordinate geometry (btoh 2-D and 3-D). The additional advantage of using the Vedic Tribhujank approach is that most of the applicants can be done without the use of bulk of the formulae and identities and the time required for study of Trigonometry can be drastically cut down. Further, Tribhujank approach can provide better accuracy and precision [illegible] technical applications. It has also been found to be more efficient for computer software.

We had a brief introduction to the Tribhujank approach in Chapter 2 on Algebra of complex number and then in straight line problems. Here we shall study some applications in trigonometric problems. A few problems of solution of triangles and inverse trigonometric functions have been presented in next two chapters, by way of illustration.

2. COMMENTS ON TRIBHUJANKS:-

1. Any common factor can be divided out.
2. These numbers essentially give us the desired angles.
3. If A is 4, 3,5 then complement of angle A, (90-A) is simply represented by 3,4,5 as the third angle of the triangle is a right angle.

Example 1: Find the Tribhujank of 2A. If(A) is 4,3,5.

A	4	3	5
A	4	3	5
2A	(4x4−3x3)	(4x3+4x3)	(5x5)
OR	7	24	25
Similarly: A−A=O	16+9	12−12	25
	25	0	25
0	1	0	1

Therefore Tribhujank of o is 1,0,1 (See Fig. 1)

The formulas for finding the Tribhujank for the double angle reduces to the simpler form. In general if

A	x	y	z
2A	$(x^2 - y^2)$	2xy	z^2

The first part can be directly obtained by using the seventh sutra meaning, 'by addition and subtraction'.

Example 2: Find the Tribhujank of A + B if A and B are complementary angles. Angle A is a irrational Tribhunak(A) 2,3

$\sqrt{13}$. therefore B will be 3,2 $\sqrt{13}$, we know the sum should be 90^o.

A	2	3	$\sqrt{13}$
B	3	2	$\sqrt{13}$
A+B	6−6	9+4	13
	0	13	13
OR	0	1	1

Therefore Tribhujank of 90^o is 0,1,1

Similarly A−B is	12	5	13

Note that the sum and difference of two irrational Tribhujank need not essentially be irrational. (See Fig.2).

Similarly Tribhujank of 180^{0} (which is double of 90^{0}) is - 1, 0, 1; of 270^{0} is 0, - 1, 1 and 360^{0} is 1, 0, 1. These results have also been summarised in figures (See Fig.3).

Example 3: Find the Tribhujank for 2A.

A	5	12	13
2A	5^2-12^2	2x5x12	13^2
	-119	120	169

The first element is negative as 2A is an obtuse angle. Therefore the base of the resultant triangle is extended in the opposite direction (See Fig.4).

contd..

Ex. 1

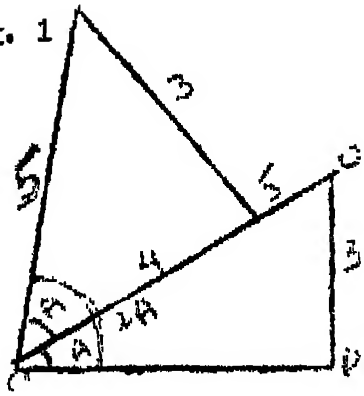


FIG. 1

Ex. 2

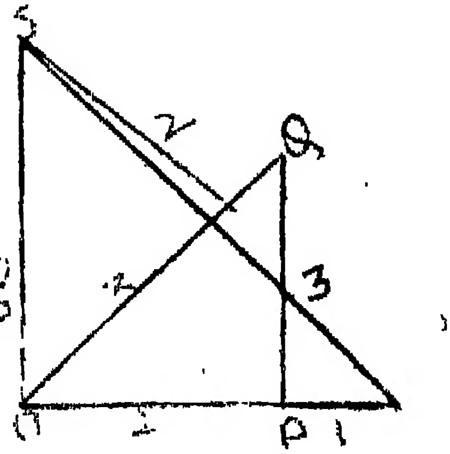


FIG. 2

Ex. 3

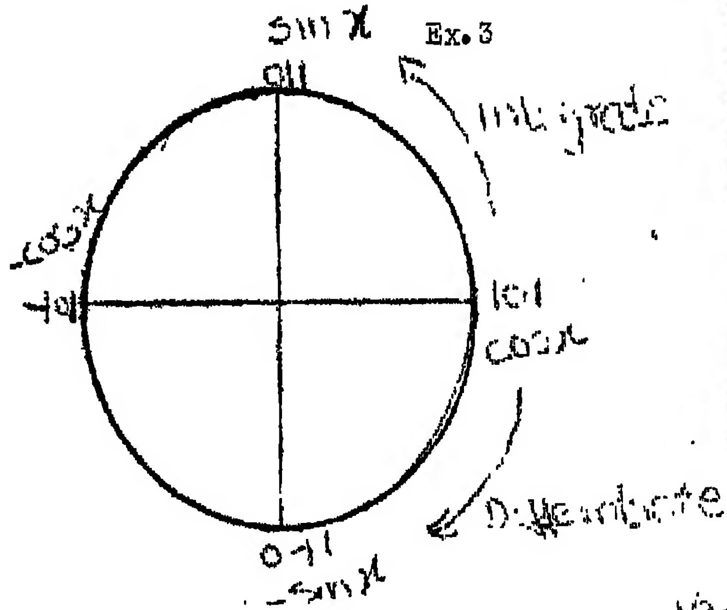


FIG. 3

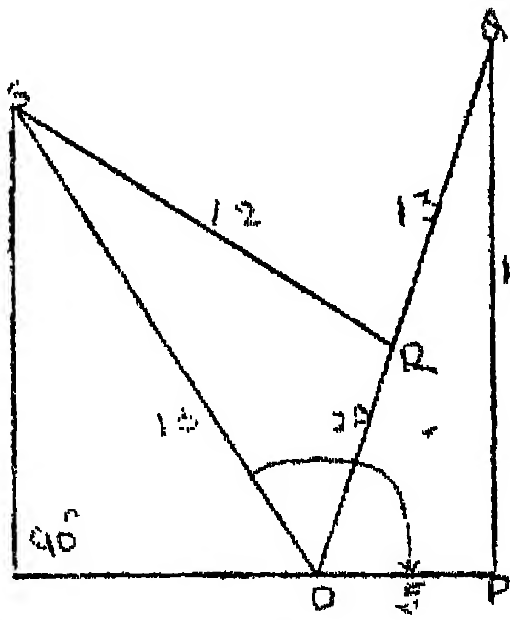


FIG. 4

Ex. 4

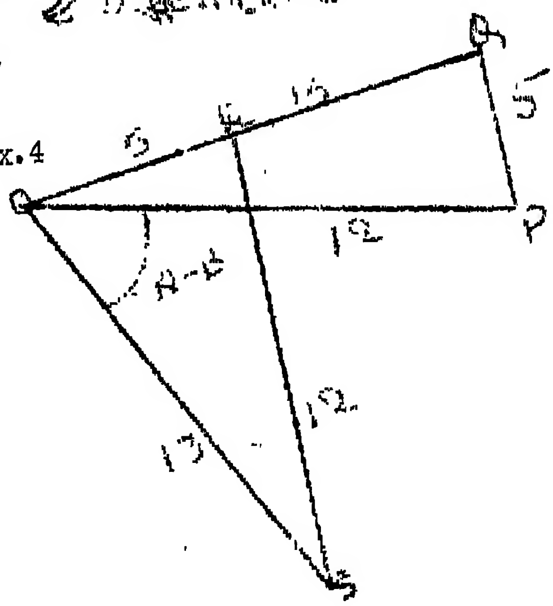


FIG. 5

Example 4: Find the Tribhujank for A-B, where (A) 12,5,13(b) 5,12,13

A	12	5	13
B	5	12	13
A-B	(60+60)	(25-144)	169
	120	119	169

The negative sign of the second element indicates that the resulting angle is negative (See fig.5)

We can easily see that the Tribhujank of any angle (-A) can be obtained by simple change of the sign of middle element.

Example 5: The tribhujank for 45^0, 30^0,60^0 etc.can be easily found from half of the sqaure and half of an equilateral triangle (See Figs.6 & 7).

45^0	1	1	$\sqrt{2}$
30^0	$\sqrt{3}$	1	2
60^0	1	$\sqrt{3}$	2

3.0 HALF ANGLE TRIBHUJANK

If (A) x,y,z then A/2 Tribhujank will have the same middle element (y). However the first element shall be addition of the first and third element (x+z) of A. Further, the third element can be computed by using the other two elements.

Example 6: Find the half angle Tribhujank of (A) 3,4,5 (Fig.8).

A	3	4	5
A/2	3+5	4	-
OR	8	4	-
	2	1	$\sqrt{5}$

Both the geometric and algebraic proofs of this are given else where(3,).

Example 7: Find the half angle Tribhujank for (A) 5,12,13. (Fig.9)

A	5	12	13
A/2	5+13	12	-
	18	12	-
	3	2	$\sqrt{13}$

The numbers in the Tribhujank do not essentially give the actual dimensions of the new triangle and shall have to be suitably modified for applications in transformation and co-ordinate geometry.

TRIGONOMETRY.

Trigonometry deals with the relations between sides and angles of traingles. We use six trigonometrical functions which aredefined as six possible different ratios formed out of the three sides of a right angled trinangle. In general let any angle A be represented by Tribhujank, x,y,z. Then we know that

contd..

Function	Abbreviated as	Formulae
Sine A	Sin A	Y/z
cosine A	Cos A	x/z
Tangent A	tan A	y/x
Cosecant A	Cosec A	z/y
secant A	sec A	z/x
Cotangent A	Cot A	x/y

The usual methods of proving identities, solving equations, computing ratios etc. in trigonometry, we have to use the appropriate formulae from a long list. But if we use the general form of Tribhujank (reducing z=1) we do not need any of the formulae. Further, even the use of the grigonometrical functions is not necessary, If A is any angle represented by the general Tribhujank p,q,i, and A can take any value between 0^0 and 360^0. The six trigonometrical functions then can be defined as cosA = p; sinA=q; tanA=q/p; cotA = p/q; secA=1/p and cosecA=1/q. We can also represent them on a single diagram. (Refer Fig. 10).

Example 8: If sinA=4/5 and sin B=12/13 find (i) tan (A+B); (ii) sin(A-B); (iii) cos (A+B): (iv) cos (2A); (v) cot(1/2A) (vi) sin(90-A) and (vii) tan (A)

The third element of each Tribhujank is obtained by using the seventh Vedic sutra, which means by addition and subtraction.

(A) - , 4,5

Base element $= \sqrt{(4+5) \quad (5-4)}$

$= \sqrt{9}$

EX. 5

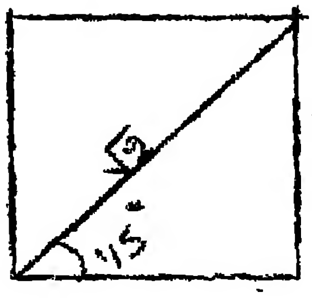


FIG. 6

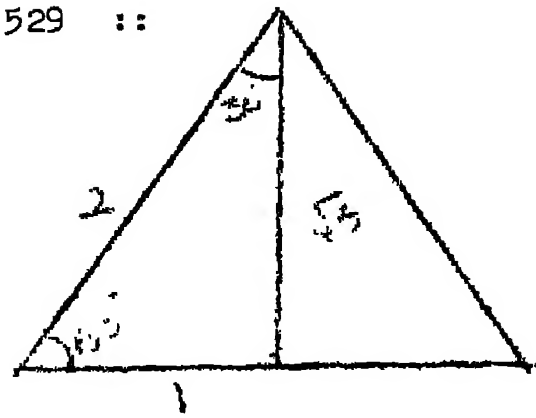


FIG. 7

EX. 6 & 7

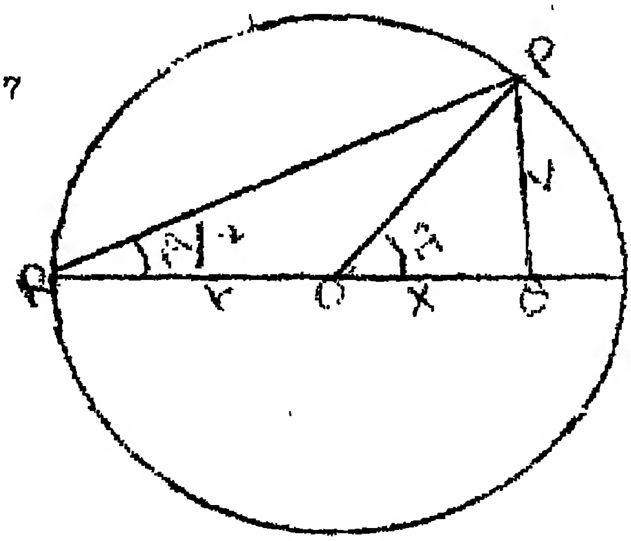


FIG. 8 & 9

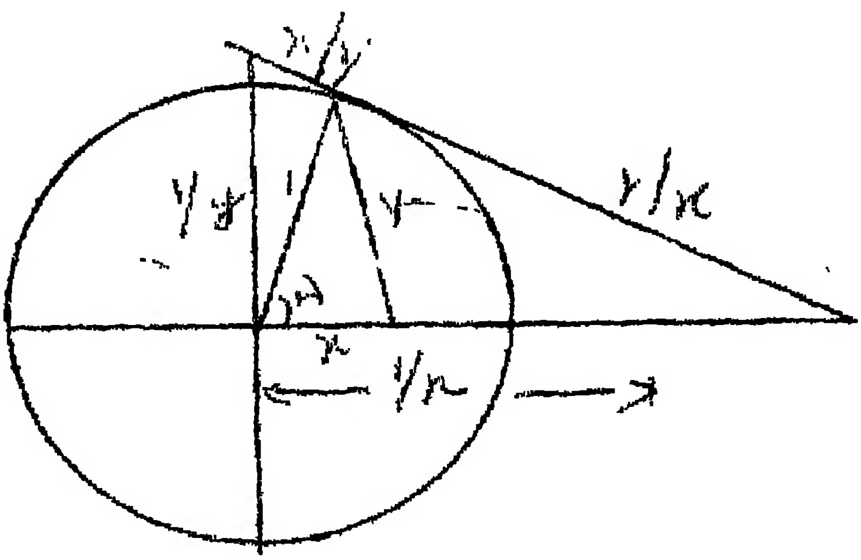


FIG. 10

Similarly (B) is 5, 12,13.

	A	3	4	5
	B	5	1..	13
(i)	A+B	(3x5−4x12)	(5x4+12x3)	13x5
		−33	56	65; tan(A+B) = $\frac{56}{33}$
(ii)	A − B	−	−16	65 sin(A−B) = $\frac{16}{65}$

It is not necessary to compute all the three elements.

(iii)	A + B			; cos(A+B) = $\frac{-33}{65}$
(iv)	2A	$3^2 - 4^2$	−	5^2
		− 7	−	25; cos2A = $-\frac{7}{25}$
(v)	A/2	3 + 5	4	−
		8	4	−
		2	1	− (A/2)=2
(vi)	90^0	0	1	1
	A	3	4	5
(vii)	90^0−A	−	3	5; sin(90^0−A) = $\frac{3}{5}$
	0^0	1	0	1
	A	3	4	5
	0^0−A	3	−4	−; tan(A) = $\frac{-4}{3}$

contd..

Example 9 : Evaluate Tribhujank for 120°, 240°, 675°, 105°, −15°,

15°.

Tribhujank for 30° is $\sqrt{3}$, 1, 2

90° is 0, 1, 1

Therefore Tribhujank for 120° is −1 $\sqrt{3}$ 2

Tribhujank for

60°	1	$\sqrt{3}$	2
180°	−1	0	1
240°	−1	$-\sqrt{3}$	2

Tribhujank for

360°	1	0	1
720°	1	0	1
45°	1	1	$\sqrt{2}$
720−45=675°	1	−1	$\sqrt{2}$

Tribhujank for

45°	1	1	$\sqrt{2}$
60°	1	$\sqrt{3}$	2
45°+60°=105°	$1-\sqrt{3}$	$1+\sqrt{3}$	$2\sqrt{2}$
45−60 =(−15)	$1+\sqrt{3}$	$1-\sqrt{3}$	$2\sqrt{2}$

Tribhujank for 15 can be obtained by two alternative ways.

60°	1	$\sqrt{3}$	2
45°	1	1	$\sqrt{2}$
60−45=15	$1+\sqrt{3}$	$\sqrt{3}-1$	$2\sqrt{2}$

OR

30°	$\sqrt{3}$	1	2
30°/2= 15°	$2+\sqrt{3}$	1	−

contd..

Example 10. Prove the following:

i) $\dfrac{\text{Sin}A + \text{Sin}2A}{\text{Cos}A - \text{Cos}2A} = \dfrac{\cot A/2}{2}$

ii) $\dfrac{\text{Cos}2B-\text{Cos}2A}{\text{Sin}2B+\text{Sin}2A} = \tan(A-B)$

iii) $\text{Sin}105^\circ + \text{Cos}105^\circ = \text{Cos}45^\circ$

Let Tribhujank for (A) b x y 1

A	x	y	1
2A	x^2-y^2	2xy	1
A/2	X+1	$y\ /(x+1)^2+y^2$	

Similarly if (B) x1, y1, 1

A	x	y	1
B	x1	y1	1
A-B	(x.x1+y.y1)	(x1.y-x.y1)	1
2B	$(x1^2-y1^2)$	2x1y1	1

(i) $\dfrac{\text{Sin}A + \text{Sin}2A}{\text{Cos}A-\text{Cos}2A} = \dfrac{y+2xy}{x-(x^2-y^2)}$

$= \dfrac{y(1+2x)}{2}$ as $y^2=1-x^2$)

$= \dfrac{y}{1-x} \quad \dfrac{1+x}{1+x}$

$= \dfrac{x+1}{y} = \cot(A/2)$

(ii) $\dfrac{\cos 2B - \cos 2A}{\sin 2B + \sin 2A} = \tan (A-B)$

$\dfrac{x1^2 - y1^2 - x^2 + y^2}{2x1.\ y1 + 2xy} = \dfrac{x1.y - x.y1}{x.x1 + y.y1}$

$\dfrac{x1^2 - x^2}{x1.y1 + xy} = \dfrac{x.x1 - x.y1}{x.x1 + y.y1.}$ $\left[\text{as } y^2 = 1-x^2\right]$

By cross multiplying, we get,

$x1^2 + y1^2 = x^2 + y^2 = 1$ (ok)

(iii) $\sin 105^0 + \cos 105^0 = \cos 45^0$

The Tribhujank of 105^0 is $1-\sqrt{3}$, $1+\sqrt{3}$, $2\sqrt{2}$, (calculated in the previous question).

Therefore,

$\dfrac{1+\sqrt{3}}{2\sqrt{2}} \quad \dfrac{1-\sqrt{3}}{2\sqrt{2}} = \dfrac{1}{\sqrt{2}} = \cos 45^0$. proved.

present method.

$\sin 105^0 + \cos 105^0 = \sin (180^0 - 105^0) - \cos (180^0 - 105^0)$

$= \sin 75^0 - \cos 75$

$= \sin 75^0 - \sin (90^0 - 75^0)$

$= \sin 75^0 - \sin 15^0$

$= 2 \cos 45^0 \sin 30^0$

$= 2 \cos 45^0 * 1/2$ *(a series of formulas have been used)

$= \cos 45^0$ proved.

Example 11: Prove the following:

(i) $\dfrac{\cos A+\sin A}{\cos A-\sin A} - \dfrac{\cos A-\sin A}{\cos A+\sin A} = 2\tan 2A$

(ii) $\dfrac{1-\cos 2A}{1+\cos 2A} = \tan^2 A$

(iii) $\sin(45^\circ+A)\sin(45^\circ-A) = (1/2)\cos 2A$

(iv) $\sec A + \tan A = \tan(45^\circ + A/2)$

(v) $\sin A \sin(60^\circ-A)\sin(60^\circ+A) = (1/4)\sin 3A$

Solutions

The Tribhujank for A

A	x	y	1
$2A$	x^2-y^2	$2xy$	1

(i) For L.H.S. $\dfrac{x+y}{x-y} - \dfrac{x-y}{x+y}$

$$= \frac{4xy}{x^2-y^2} = 2\tan 2A \quad \text{R.H.S. Proved}$$

(ii) For L.H.S.

$$\frac{1-(x^2-y^2)}{1+(x^2-y^2)} = \frac{y^2}{x^2} = \tan^2 A \quad \text{R.H.S. Proved.}$$

(iii) The Tribhujank for

45°	1	1	$\sqrt{2}$
A	x	y	1
$45^\circ+A$	$x-y$	$x+y$	$\sqrt{2}$
$45^\circ-A$	$x+y$	$x-y$	$\sqrt{2}$

contd..

For L.H.S.

$\sin(45^\circ + A)\ \sin(45^\circ - A) = \left(\frac{x+y}{\sqrt{2}}\right)\ \left(\frac{x-y}{\sqrt{2}}\right)$

$= \frac{x^2 - y^2}{2} = (\cos 2A)/2$ R.H.S.

(iv) The Tribhuj nk for

A	x	y	1
A/2	x+1	y	–
45°	1	1	$\sqrt{2}$
45°+A/2	(x+1−y)	(y+x+1)	

For L.H.S. $\sec A + \tan A = \frac{1}{x} + \frac{y}{x}$

$= \frac{1+y}{x} =$ R.H.S.

R.H.S. $= (y + x + 1)/\ (x-y+1)$

Cross multiply and simplify

$x^2 + y^2 = 1$

$1 = 1$ Proved.

(v) The Tribhuj nk for

60°	1	$\sqrt{3}$	2
A	x	y	1
60°+A	$(x-\sqrt{3}y)$	$(x\sqrt{3}+y)$	2
60°−A	$(x+\sqrt{3}y)$	$(x\sqrt{3}-y)$	2
2A	(x^2-y^2)	2xy	1
A	x	y	1
3A	(x^3-3xy^2)	$(3x^2y-y^3)$	1

contd..

For L.H.S.

sinA sin (60−A) sin (60+A)

$$= y \left(\frac{\sqrt{3}\cdot x - y}{2}\right) \left(\frac{\sqrt{3}\cdot x + y}{2}\right)$$

$$= \frac{3x^2 y - y^3}{4} = \frac{1}{4}\sin 3A \quad \text{R.H.S.}$$

4.3 Trigonometrical Equations:

Example 12: Solve the following equations in Tribhujank form:

(i) $\cos A = 1/\sqrt{2}$

(ii) $\cot B = 1/\sqrt{3}$

(iii) $\cot A + \tan A = 2\sec A$

Solutions:

(i) If $\cos A = 1/\sqrt{2}$,

then Tribhujank for A is (A) 1, − , $\sqrt{2}$

Therefore 1, 1, $\sqrt{2}$ and 1, −1 $\sqrt{2}$ are two solutions.

(ii) If $\cot B = 1/\sqrt{3}$

then Tribhujank for B is (B) 1, $\sqrt{3}$ −

Therefore, 1, $\sqrt{3}$ 2 and 1 $\sqrt{3}$, − 2 are the ~~two~~ solutions.

(iii) Let the general Tribhujank for A) be x, y, 1

Then

cosA + tanA = 2· cA

$$\frac{x}{y} + \frac{y}{x} = \frac{2}{y}$$

$$\frac{x^2 + y^2}{xy} = \frac{2}{y}$$

$$x = 1/2$$

Therefore th Tribhujank for

A) 1 - 2

$1 \pm \sqrt{3}$, 2 . . th . two solutions.

5. BEEJANK:

In Vedic Mathematics B ejganit is th original name for Algebra. B j essentially m ans the sedd and ganit m ans mathematics. Ther fore, Beejganit should ess ntially b th seed of different fac ts f Math matics. This is quite clear and obvi us if n studies Math mantics the Ancient V dic way. Now w shall introduce ourselves to the B jank of diff r nt Tribhujanks, that is th s d nu b rs. W shall study Beejank through th use of B jganit (Alg bra), although the whol thing c n also b introduc d through Vedic geom try.

Let p and q be th first and second elements of general Tribhujank, r pres nting a half angle A/2.

(A/2) p, q —

p, q, $\sqrt{p^2 + q^2}$

Ther for

contd..

(A) $p^2 - q^2,\ 2pq,\ p^2 + q^2$ (1)

Here all the elements of Tribhujank for (A) shall be rational if p,q are rational. Further, p and q can be used to generate a perfect Tribhujank of (A) (with three elements of (A)) as such p,q, are called as the Beejanks, (seed numbers) of Tribhujank of angle.

(A) Frequently q may be equal to unity and in that case only Beejank p is sufficient to completely define the Tribhujank.

(A) $p^2 - 1,\ 2p,\ p^2 + 1$

If If we substitute p equal to 3,4,5,6,7 etc. we get the following Tribhujank.

TABLE I

TRIBHUJANK

Beejank	Raw	Final	Middle element
3	$3^2 - 1$, 2x3, 3^2+1	4,3,4	3
	8, 6, 10		
4	15,8,17	15,8,17	2 x 4
5	24,10,26	12, 5, 13	5
6	35,12,37	35,12,37	2 x 6
7	48, 14, 50	24, 7, 25	7
8	63, 16, 65	63, 16, 65	2 x 8
9	80, 18, 82	40, 9, 41	9
	etc.		

Therefore we can see that a long list of Tribhujank can be

built up by using B j nk p only. In g n ral, we use two Baujanks p, q. In this case since q is 1 we are adding and subtracting the square of q (which is equal to 1) to obtain the first and third element using the Vedic sutra No.7. The middle element is obtained by using the upsutra Dwandw yoga (No.14) (Duplex of p, q, is 2pq, the middl lement.)

5.1 Tribhujank Pattern :

If we carefully study the pattern of the Tribhujanks being built up by using the whole number Baujanks from 3 onwards, we can clearly observe that we are getting two distinct groups of Tribhujanks.

Group 1: It consists of the Tribhujanks obtained from even Baujank (4;6;8 etc) and they have following distinct properties

i) The middle element is double the Baujank (2*4=8, 12,16 etc)

ii) The third element is always 2 more than the first (15+2 = 17; 35+2=37 etc).

iii) Further, two times the sum of the first and third element is equal to the square of the middle element.

$$2(15 + 17) = 64 = 8*8$$

$$2(35 + 37) = 144 = 12*12 \text{ etc.}$$

Group 2: The Tribhujanks computed by using the odd number Baujanks (3;5;7 etc) and the three elements of this group exhibit the following properties:

i) The middle element is the Baujank (3;57 etc)

ii)The third element is always one more than the

iii) Further, the sum of first and the third element is equal to the square of the middle element.

(4 + 5) = 9 = 3*3

(12 +13) = 25 = 5*5 etc.

OR

In other words we can say that both the groups have very similar properties except that we have the factor of 2 present for even Beejank and this factor becomes 1 for odd Beejanks. The reason of this is quite obvious if we carefully study the raw and the final elements in the Table I, of Tribhujank. (as we have already divided by 2 the raw elements of odd Beejank).

These properties of Tribhujanks are very useful and can be used in two alternative ways. Firstly we can build up the three elements of any Tribhujank from the Beejank. Secondly, we can check the correctness fo three elements computed by using the Beejank formulae

$(p^2 - 1, 2p, p^2 + 1)$.

It will be interesting to point out that Vedic Maths provides two alternative approaches which are independent of each other.

Example 13: Compute the Tribhujank whose Beejank is

(i) 7 (ii) 13 (iii) 27 (iv) 36 (v) 31

Solutions:

(i) p = 7

Tribhujank 7*7-1, 2*7*1, 7*7+1

48 14 50

Therefore 24, 7, 25

Check: (i) 25 −24 = 1

(ii) 7*7 = 49 = 24 + 25 Okay

(ii) p = 13

Tribhujank 13*13−1, 2*13*1, 13*13+1

84 13, 85

Check: (i) 85−84 = 1

(ii) 13*13 = 169 = 84 + 85 Okay

(iii) p = 27

Tribhujank 27*27−1, 2*27*1, 27*27+1

364 27 365

Check: (i) 365−364 = 1

(ii) 27*27 = 729 = 364 + 365

(iv) p = 36

Tribhujank 36*36 − 1 2*36*1, 36*36+1

1295 72 1297

Check: (i) 1297−1295 = 2 because Baujank is even

(ii) (72*72)/2 = 5184/2 = 2592

= 1295 + 1297 Okay

(v) p= 31

Tribhujank 31*31−1, 2*31*1, 31*31+1

960 62 962

Therefore 480 31 481

Check: (i) 481−480 = 1

(ii) 31*31 = 961= 480 + 481 Okay.

5.2 Baujank as Tribhujank:

It is very useful to learn that the Baujanks themselves are also the Tribhujank of half angle. If p is the only Baujank the corresponding Tribhujank is p, 1, − or p,1, $\sqrt{1+p^2}$

Further if p, q are the Beejanks the Tribhujank is

p,q - or

p, q $\sqrt{p2 + q^2}$

5.3 Two Beejanks:

In computing the Tribhujank of table 1, we have used only integer Beejanks starting with 3, we can also have a set of Tribhujanks if we take Beejank as 2.5, 3.5 etc. or in other words Beejank is 2.5,1.

OR 5,2 here p= 5 and q=2. This will give as another set of Tribhujanks. In addition to those already obtained in Table I. However, the general formulae (1) for computing the Tribhujank for the given Beejanks remains the same.

Example 14: Compute the Tribhujank if (i) p, q=12,5 (ii) p,q = 7,2(iii) p,q = 9,2

Solution:

(i) p,q = 12,5

Tribhujank 12*12 - 5*5, 2*12*5, 12*12+5*5

119, 120, 169

(ii) p,q = 7,2

Tribhujank 7*7 - 2*2, 2*7*2, 7*7+2*2

+45, 28, 53

(iii) p, q = 9,2

Tribhujank 9*9-2*2, 2*9*2, 9*9+2*2

77 36 85

5.4 Computing B ..j. k fr. Tribhujank:

The B. jank of any Tribhujank can b. easily obtained by using any of the two alternativ methods.

(i) Since we know that B. jank itself is a Tribhujank of half angle, therefore using the half angle formulae of Tribhujank p shall b. the sum of the outer two elements and q shall b the middle element itself.

Example 15: For Tribhujank 4,3,5 B jank (p,q) shall be 4+ 5, 3 i.e. 9,3;3,1.

(ii) The B. jank can also b obtained by using Vedic sutra (No.7) by addition and subtraction p shall be the square root of the addition of the first and the third element of Tribhujank and q shall be the square root of the difference of the first and the third element of Tribhujank.

Example 16: (i) For Tribhuj nk 4,3,5

Beejank (p,q) shall be $\sqrt{5+4}$ $\sqrt{5-4}$ i.e. 3,1

(ii) For Tribhujank 77, 36, 85

By first method 85 + 77, 36

i.e. 162, 36, 9,2

(iii) For Eribujank 399, 40, 401

By first method 401 + 399, 40

i. 800, 40; 20, 1

(iv) For Tribhujank 45, 28, 53

By first method 53 + 45, 28

i. . 98, 28, 7, 2

(v) For Tribhujank 480, 31,481

By first method 480 + 481, 31

i.e. 961, 31, 31,1

Discussion:

(i) We can see that the Beejank for any Tribhujank has much smaller numbers, as such it is always much more convenient to operate the Beejanks.

(ii) As usual Vedic Maths provides us two alternative ways and hence the second method can be used for checking.

5.5 Beejank of Beejank:

As demonstrated in the Katyayan's sulbha sutra it is quite conveniently possible to find out the Beejank of the Beejanks themselves, as Beejank itself is expressing a Tribhujank.

Example 17: The Tribhujank 119, 120, 169 has Beejank 12, 5. This itself represents a Tribhujank 12, 5, 13, as such the Beejank for 12, 5, 13 shall be 12+ 13 5 i.e. 5,1.

5.6 Beejank for Supplementary Tribhujank:

The Beejank for supplementary Tribhujank is obtained directly by using the Paravartaya Yojyet sutra (No.4) i.e. by transposing the Beejank.

Example 18: If p, q = 12, 5

By transposition p, q (Sf) = 5,12

5.7 Beejank for Complementary Tribhujank are obtained by using the sutra Sankalana Vyavakalana (No.7) meaning, by addition and subtraction'.

contd..

For Beejank p, q(CT) = (p+q), (p-q)

Example 19: If p, q= 12,5

pq(CT = (12+5), (12-5) = 17-7

5.8 Beejank for O, 90^0, 120^0 and 270^0 :

The Beejank for these shall be 1,0; . . 0,1; 1, -1 or ∞,1,0, - 1 respectively. Further, we can clearly see that by simply looking at the magnitude and sign of Beejank p itself we can find out the quadrant of the angle whose Beejank is represented by p.

1st quadrant From 0^0 to 90^0 Beejank p varies from ∞ to 1.

2nd quadrant from 90^0 to 180^0 Beejank p varies from 1 to 0

3rd quadrant from 180^0 to 270^0 Beejank p varies from 0 to -1.

4th quadrant from 270^0 to 360^0 Beejank p varies from -1 to -∞

5.9 Beejank operations:

As the pair of Beejanks essentially represents a Tribhujank of half angle, therefore we can add and subtract Beejanks by using the Tribhujank procedure and further the sum of the Beejanks shall also be the Beejank for the sum of the Tribhujank. This property greatly simplifies the addition and subtaction process of Tribhunank as Beejank are much smaller in magnitude.

Example 20: Beejank of (A) 12, 5, 13 shall be 5,1 and Beejank of (B) 4,3,5 shall be 3,1. Addition of Beejanks gives

p(A)	5,1
p(B)	3,1
p(A)+p(B)	14,8 = 7,4

CHAPTER-14

SOLUTION OF TRIANGLES .... USING VEDIC MATHS.

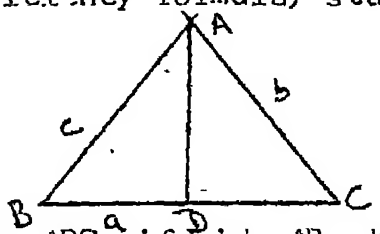
The solution of plane triangles can be very conveniently done by using the Vedic Tribhujank concept.

Let A,B,C be the three vertices of a triangle and a,b,c, be the sides opposite to these vertices. There are only three useful formulae which we use for the solution of planetriangles in addition to the Tribhujank formulae.

1. Kone Nyunena formula (Angle deficiency formula) states that

$a^2 + b^2 - c^2 = 2a*DC$

Where DC is the partial base.



2. Anurupyena formula: In triangle ABC, if side AB= t and AC=T, and the Tribhujank for angles B and C are (B) q,r,s and (C) Q, R, S then we can easily see that AD=t*r/s and also AD= T*R/S.

Therefore t*r/s = T*R/S this is the Anurupyena formula.

3. Beejank formula : For the traiangle ABC

(B + C) / (b-c) = p(B-C)/p(B+C).

Example 1: If a, b,c are 6, 5,4 Find B.

Solution:By Nyunena formula

$a^2 + c^2 - b^2 = 2a*BD$

Therefore BD = L.H.S./(2*BC)

= 4*4 + 6*6 - 5*5/2*6

= 9/4.

therefore angle B = 9/4 , - ,4

= 9, - 16

If so desired this can further be converted into angle.

Example 2: If a and c ar. 21 & 10 and angle B is 3,4,5 compute b.

Solution: Since Tribhuja of B is 3,4,5 therefore BD= 10*3/5 = 6 using Nyuna formula.

10*10 + 21*21 - b*b + 2*6*21

therefore b = 17

Example 3: If b=20 and, angle B and C are 12,5,13, and 3,4,5 respectively. Compute C.

Solution: Using the Anurupyana formula C*5/13 = 20*4/5

Therefore C = 41.6

Example 4: If b = 2 and c = 6 angle c=3,4,5 compute angle B.

Solution: Let the Tribhujank of angle B be q,r,s, then,

using the Anurupyana formula

6*r/s = 2*4/5

Therefore r/s = 4/15

Hence angle B = -, 4,15

Example:5 If b, c are 2,3 and angle A is 4,3,5 Find B.

Solution: P(A) = 3(Beejank)

since (B+C) is supplementary angle of A.

p(B+C) = 1/3

Using Beejank formula

p(B-C)/p(B+C)= (b+c)/(b-c)

p(B-C)=(3+2)/[(3-2) * 3)]

=5/3

p(B+C)	1	3	-
p(B-C)	5	3	-
adding p(2B)	-4	18	-

Therefore B=2,9(Since p(2B) = B)

We can also compute angle C by subtraction.

Example 6: Exampl 3 can also b done by using th Beejank formulae. Although this approach is unnecessarily long.

p(B) = 5, p(C) = 2 P(B – C) = 11, –3.

Therefore p(B+C) = 9,7 and Using Beejank formula

(11/-3)* (7/9) = (20+c)/20-C)

77/27 = (c + 20)/c-20)

50c = 2080

c = 41.6

Example 7: In a right angle triangle if a, b are 41,13 compute angle B.

Solution: Here (B) 41,13, we know that 41, 13 are the Beejank of a Triangle containing an angle 2B. So we can find 2B, 41,13 is close to 3,1, therefore:

41	13
3	1

Subtracting 136 –1 = 68, –1

Therefore using table I1

2B = .6435 - 2/68

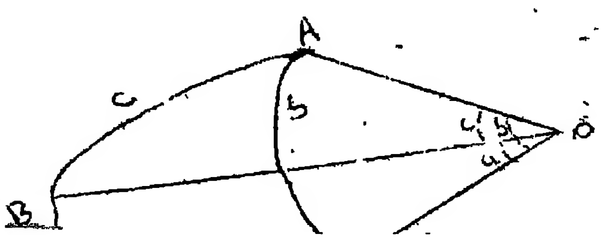
= 6141

Therefore B = .3070

SPHERICAL TRAINGLES

We know spherical triangle A,B,C is a triangle on the surface of a sphere, the sides being arcs of great circles. The three sides are expressed as angles, they make at the centre of the sphere.

Further details are available elsewhere (Ref.3,5)



CHAPTER-15

Inverse functions: Using Vedic Mathematics:

It is interesting to observe that handling of inverse functions using the Vedic Tribhuja approach, involves absolutely nothing new. A few illustrations are given here to demonstrate the ease and simplicity provided by the Vedic methods (Ref.3,5)

Example: Evaluate (i) $\cot^{-1}(2) + \cot^{-1}(3)$,

(ii) $2\tan^{-1}(3) + \cot^{-1}(2)$

(iii) $\sin^{-1}(3/5) + \tan^{-1}(2)$

We are required to add the angles represented by the individual inverse functions.

(i)

A	2	1	5
B	3	1	$\sqrt{10}$
A+B	5	5	–
	1	1	$\sqrt{2}$

Therefore A+B = 45°

(ii)

A	1	3	$\sqrt{10}$
2A	−8	6	10
B	2	1	$\sqrt{5}$
2A + B	−22	4	–
	−11	2	–

Therefore $2\tan^{-1}(3) + \cot^{-1}(2) = \tan^{-1}(-2/11)$

(iii)

A	4	3	5
B	1	2	$\sqrt{5}$
A+B	−2	11	$5\sqrt{5}$

Therefore $\sin^{-1}(3/5) + \tan^{-1}(2) = \tan^{-1}(-11/2)$

CHAPTER - 16

STATISTICS

Introduction: We have already seen the applications of Vedic mathematics in solving the problems of coordinate geometry and trigonometry. Vedic Maths provides very simple quick and easy methods for combined operations of arithmatic. Brief introduction to arithmatical operations has already been presented in a previous chapter. Statistics requires a series of combined operations of arithmatic for solving different problems. Vedic methods can be used to provide speed and simplicity. Applications of the Vedic procedures are illustrated with the help of a few examples.

EXAMPLE 1: The ages of all the male inhabitants of a village were recorded and the following frequence table was obtained:

Age (years)	No.of persons
0 - 5	12
5 - 10	18
10 - 20	16
20 - 30	19
30 - 40	14
40 - 50	11
50 - 60	04
60 - 80	03

Obtain the mean age per male inhabitant.

Solution:

In this example we first compute the total number of persons by adding the number of persons in each group. This can be very efficiently done by using Vedic upsutra Shudha (Ref.2,3, & 6)

Age (Years)	No. of persons f_i	Mean Value x_i
Ø - 5	12	2.5
5-1Ø	18	7.5
1Ø - 2Ø	16	15.Ø
2Ø - 3Ø	19	25.Ø
3Ø - 4Ø	14	35.Ø
4Ø - 5Ø	11	45.Ø
5Ø - 6Ø	4	55.Ø
6Ø - 8Ø	3	70.Ø

$$\sum f_i = 97$$

Mean value of x_i is simply the average of the two limits of age, in each interval.

For computing the sum of the products of f_i and X_i we use the combined operations of multiplication and addition using the Urdhvasutra.

In step 1 we consider the vertical products of the right most digits.

In Step 2 we consider the cross products of two digits of left hand side etc.

$\sum f_i . x_i$ = 1Ø/11Ø/19Ø/5Ø = 2295.Ø

Step 1: 5x2 + 5x8 + Øx6 +.........+Øx3 = 5Ø

Step 2: (2x2+1x5) + (7x8+5x1) + (5x6 + Øx1)
+(5x9+Øx1) + (5x4 + Øx1) + (5x1+Øx1)
+(5x4+Øx0) + (Øx3+ØxØ)

= 9+ 61 + 3Ø + 45 + 2Ø + 5 + 2Ø + Ø = 19Ø

Step 3: 2x1 + 7x1 + (1x6+5x1) + (2x9+5x1)
+(3x4+5x1) + (4x1+5x1) + (5x4+5xØ) + (7x3+ØxØ)

= 9 + 11 + 23 + 17 + 9 + 2Ø + 21 = 11Ø

Step 4 : $0 + 0 + 1x1 + 2x1 + 3x1 + 4x1 + 5x0 + 7x0$

$= 1 + 2 + 3 + 4 + 0 + 0 = 10$

For division we can use the Vedic Dhvajank method of straight division (Ref 2 & 3). In this case since divisor (97) consists of big digits, even Nikhilam method of division can be very efficiently utilised.

$$\bar{X} = \sum_{i=1}^{m} \frac{f_i \cdot x_i}{f_i} = \frac{2295}{97}$$

$$\begin{array}{r|llll} 9 & {}^{7}/\ 2 & 2_4 & 9_8 & 5 \\ \hline & 2 & 3 & 7 & \end{array}$$

$22 \div 9 = 2$, R4; $7x2 = 14$

$49 - 14 = 35$; $35 \div 9 = 3$ R8;

$7x3 = 21$, $85 - 21 = 64$; $64 \div 9 = 6$ R10 ;

$7x6 = 42$, $100 - 42 = 58$; $58 \div 9 = 6$ R4 etc.

$\bar{X} = 23.66$

The mean age is 23.7 years approximately

EXAMPLE 2: The postal expenses on the letters despatched from an office on a given day resulted in the following frequency distribution.

Postage (P)	No. of letters
15	47
30	33
35	56
60	41
70	25

Find the mean postage per letter. Convert the postal charges in rupees and then calculate the mean postage per letter.

Postage (x_i) (paise)	No. of letters (f_i)
15	47
30	33
35	56
60	41
70	25
	$\sum f_i = 202$

Once again we can use the Vedic Shudha method of addition. The summation of the products can be directly obtained by using the combined operations of arithmetic as outlined in the following three steps.

$$\sum f_i x_i = (15\times47) + (30\times33) + (35\times56) + (60\times41) + (70\times25)$$

Step-1: $(5\times7) + (0\times3) + (5\times6) + (0\times1) + (0\times5) = 65$

carry digit 6

Step 2: $6\ (\text{carry}) + (1\times7+5\times4) + (3\times3+0\times3) + (3\times6+5\times5) + (6\times1+0\times4) + (7\times5+0\times2) = 126$

carry over = 12

Step 3: $12\ (\text{Carry}) + (1\times4) + (3\times3) + (3\times5) + (6\times4) + (7\times2) = 78$

$$\sum f_i x_i = 7865$$

$$\therefore \bar{x} = \sum_{i=1}^{n} \frac{f_i x_i}{f_i} = \frac{7865}{202}$$

In actual working we do not have to write down these steps

contd.

and most of the working can be done mentally.

The division is carried out directly in one line by using the Vedic Dhwajank method of straight division (Ref. 2 & 3)

$$_{2}\varnothing 2 \; / \; 7\;{}_{1}8\;{}_{\bar{0}}8\;{}_{\bar{0}}5$$

$$39.\varnothing\bar{6}$$

$$= 39.94$$

$7 \div 2 = 3 \; R1;$

$\varnothing \times 3 = \varnothing,\; 18-\varnothing = 18;\; 18 \div 2 = 9,\; R\;\varnothing$

$\varnothing \times 9 + 2\times 3 = 6,\; \varnothing 6 - 6 = \varnothing;\; \varnothing \div 2 = \varnothing, R\;0$

$\varnothing \times \varnothing + 2\times 9 = 18, \varnothing 5 - 18 = \overline{13} \quad \overline{13} \div 2 = \bar{6}\; R\; \bar{1}$ etc.

$\bar{X}$ = 38.94 Paise per letter

$\bar{X}$ = .39 Rupees per letter.

EXAMPLE 3: The scores of batsman A were 38, 7Ø 48, 34, 42, 55, 63, 46, 54, 44 find the variance.

Solution:

x_i	$(x_i)^2$
38	1 4 4 4
7Ø	4 9 Ø Ø
48	2 3 Ø4
34	1 1 5 6
42	1 7 6 4
55	3 Ø 2 5
63	3 9 6 9
46	2 1 1 6
54	2 9 1 6
44	1 9 3 6

$\sum x_i$ = 4 9 4 $\quad \sum (x_i)^2$ = 2 5 5 3 Ø

For $(x_i)^2$; we take duplex of numbers (Refer chapter 2 ORAVM Pushp 3 for details)

$$(38)^2 = 9/4^8/6^4 = 1444$$

$$(70)^2 = 49/0/0 = 4900$$

$$(48)^2 = 16/6^4/6^4 = 2304$$

$$(44)^2 = 16/3^2/1^6 = 1936$$

$$\frac{(\sum x_i)^2}{n} = \frac{(494)^2}{10}$$

$$= \frac{16/7^2/11^3/7^2/1^6}{10}$$

$$= \frac{244036}{10} = 24403.6$$

$$\sum(x_i)^2 - \frac{(\sum x_i)^2}{n}$$

$$= 25530 - 24403.6$$

Variance $= \sigma^2 = \frac{1126.4}{10}$

$$\sigma^2 = 112.64$$

EXAMPLE 4: The measurements (in mm) of the diameter of the heads of 107 screws gave the following frequency table.

Diameter	Frequency
33-35	17
36-38	19
39-41	23
42-44	21
45-47	27

Calculate the mean head diameter per screw.

Solution:

Diameter	Mid value	Frequency(f_i)
33-35	34	17
36-38	37	19
39-41	4Ø	23
42-44	43	21
45-47	46	27
		$\sum f_i = 1Ø7$

For $\sum f_i x_i$ =

Step 1: 4x7 + 7x9 + Ø x 3 + 3x1 + 6x7

=28 + 63 + Ø + 3 + 42 = 136

Carry over 13

Step 2: 13 (Carry) + (3x7+4x1) + (3x9+7x1)

=(4x3+Øx2) + (4x1+3x2) + (4x7+6x2)

= 13 + 25 + 34 + 12 + 1Ø + 4Ø = 234

Carry over 13

Step 3: 13 (carry) + 3 x 1 + [illegible] x1 + 4x2 + [illegible]x2 + 4x2

= 13 + 3 + 3 + 8 + 8 = 43

$\sum f_i x_i$ = 4346

1Ø7 / 4 3 4 6 Ø

4 3 $\bar{2}$ $\bar{4}$ = 41.$\bar{4}$ = 4Ø.6

4 ÷ 1 = 4 R O; Ø x 4 = Ø, Ø3 - Ø = Ø3

Ø3 ÷ 1 = 3 R O; Ø x 3 + 7x4 = 28

Ø4 - 28 = $\bar{2}$ $\bar{4}$ $\bar{2}$ $\bar{4}$ ÷ 1 = $\bar{2}$ $\bar{4}$ R O

....

APPENDIX-II

Workshop for Development of Teachers' Guide in Mathematics for Class XI.

VENUE : The Ramanujan Institute for advanced study in Mathematics, Univ. of Madras, Madras.

DURATION: 21 - 27 August 1989.

LIST OF PARTICIPANTS

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21.	Prof. K.V.Rao	DESM - NCERT, Sri Aurobindo Marg, New Delhi-110016.
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APPENDIX - III

Workshop for finalization of Teachers' Guide in Mathematics for Class XI.

VENUE : S I E R T, Udaipur, Rajasthan

DURATION: 15 - 24 January, 1990.

List of Participants

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9.	Shri Ganpat Burad	SIERT, Udaipur, Rajasthan.
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